

DISPERSION FOR THE WAVE EQUATION INSIDE STRICTLY CONVEX DOMAINS I: THE FRIEDLANDER MODEL CASE

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ABSTRACT. We consider a model case for a strictly convex domain $\Omega \subset \mathbb{R}^d$ of dimension $d \geq 2$ with smooth boundary $\partial\Omega \neq \emptyset$ and we describe dispersion for the wave equation with Dirichlet boundary conditions. More specifically, we obtain the optimal fixed time decay rate for the smoothed out Green function: a $t^{1/4}$ loss occurs with respect to the boundary less case, due to repeated occurrences of swallowtail type singularities in the wave front set.

1. INTRODUCTION

Let us consider solutions of the linear wave equation on a manifold (Ω, g) , with (possibly empty) boundary $\partial\Omega$:

$$(1.1) \quad \begin{cases} (\partial_t^2 - \Delta_g)u(t, x) = 0, & x \in \Omega \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \\ u(t, x) = 0, & x \in \partial\Omega, \end{cases}$$

where Δ_g denotes the Laplace-Beltrami operator on Ω .

When dealing with the Cauchy problem for nonlinear wave equations, one starts with perturbative techniques faces the difficulty of controlling the size of solutions to the linear equation in terms of the size of the initial data. Of course, one has to quantify this notion of size by specifying a suitable (space-time) norm. It turns out that, especially at low regularities, mixed norms of type $L_t^p L_x^q$ are particularly useful. Moreover, the arguments leading to such estimates turn out to be useful when considering spectral cluster estimates, which are of independent interest (see [15]).

On any smooth Riemannian manifold without boundary, the following set of so-called Strichartz estimates holds for solutions of the wave equation (1.1) (for $T < \infty$)

$$(1.2) \quad \|u\|_{L^q(0,T)L^r(\Omega)} \leq C_T \left(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)} \right),$$

where, if d denotes the dimension of the manifold, we have $\beta = d(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$ (which is consistent with scaling) and where the pair (q, r) is wave-admissible, i.e.

$$(1.3) \quad q \geq 2, \quad \frac{2}{q} + \frac{d-1}{r} \leq \frac{d-1}{2} \quad (q > 2 \text{ if } d = 3 \text{ and } q \geq 4 \text{ if } d = 2).$$

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When equality holds in (1.3) we say that the pair (q, r) is sharp wave-admissible. Here H^β denotes the L^2 Sobolev space over Ω . If this holds for $T = \infty$ we say there are global Strichartz estimates. Such inequalities were long ago established for Minkowski space (flat metrics) and can be generalized to any (Ω, g) where g is smooth, at least locally in time (thanks to the finite speed of propagation).

The canonical path leading to such Strichartz estimates is to obtain a stronger, fixed time, dispersion estimate, which is then combined with energy conservation, interpolation and TT^* arguments to obtain (1.2). Let us denote by $e^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}$ the half-wave propagators in flat space, and $\psi \in C_0^\infty([0, \infty[)$. The following dispersion inequality holds:

$$(1.4) \quad \|\psi(-h^2\Delta_{\mathbb{R}^d})e^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}\|_{L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)} \leq C(d)h^{-d} \min\{1, (h/|t|)^{\frac{d-1}{2}}\}.$$

Our aim in the present paper is to obtain these estimates inside domains. In fact, [9] outlines a roadmap to prove such a dispersion estimate, on a finite time interval, for solutions of (1.1) inside a strictly convex domain (Ω, g) of dimension $d \geq 2$. A complete description of the geometry of the (semi-classical) wave front set is provided for the solution to (1.1) with initial data $(u_0, u_1) = (\delta_a, 0)$, where $a \in \Omega$ is a point sufficiently close to the boundary (depending on the scale h). This wave front set has caustics developing in arbitrarily small times and this induces a loss of $1/4$ in (1.4) for the $h/|t|$ factor.

In the present work, we aim at completing the roadmap by constructing a suitable parametrix for such a solution and then proving dispersion for the approximated solution. It should be noted that parametrices have been available for the boundary value problem for a long time, see [11, 10, 4], as a crucial tool to establish propagation of singularities for the wave equation on domains. However, while efficient at proving that singularities travel along the (generalized) bi-characteristic flow, they do not seem strong enough to obtain dispersion, at least in the presence of gliding rays. In the outside of a strictly convex obstacle (no gliding rays), the Melrose-Taylor parametrix was utilized in [14] to prove Strichartz estimates hold as in the \mathbb{R}^d case. All other positive results ([2] and references therein) rely instead on reflecting the metric across the boundary and considering a boundary less manifold with a Lipschitz metric across an interface, and then using the machinery originally developed for low regularity metrics [13, 17] and spectral cluster estimates [15]. Such constructions do away with multiply reflected rays by suitable microlocalizations: one ends up working on a possibly very small time interval, depending on the incidence of the wave packet under consideration, such that all corresponding rays are only reflected once. Summing these intervals induces (scale-invariant) losses, which get worse with dimension; while Strichartz estimates are obtained in a more direct way in [2], one can observe that the corresponding dispersion estimate would have at most $1/t$ decay for $d \geq 4$, as the argument is blind to the full dispersion which should occur in tangential directions. On the other hand, negative results were obtained in [6, 5], where a special solution is constructed, propagating a cusp across multiple reflections and providing a counterexample to the sharp Strichartz estimates (1.2), for $r > 4$. This special solution is constructed via a microlocal parametrix which utilizes the Melrose one, and our present construction generalizes this

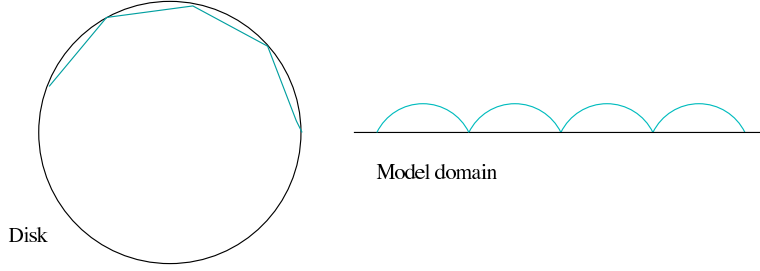


FIGURE 1.1. Light rays in the Friedlander model

special example while retaining most of its useful features.

Before stating our main result, we briefly introduce the Friedlander's model domain of the half-space $\Omega_d = \{(x, y) | x > 0, y \in \mathbb{R}^{d-1}\}$ with Laplace operator given by

$$\Delta_g = \partial_x^2 + (1+x)\Delta_y.$$

By rotational symmetry, we will eventually reduce to the two-dimensional case Ω_2 .

Remark 1.1. For the metric $g = dx^2 + (1+x)^{-1}dy^2$, the Laplace-Beltrami operator is $\Delta_{g,0} = (1+x)^{1/2}\partial_x(1+x)^{-1/2}\partial_x + (1+x)\Delta_y$ which is self adjoint with the volume form $\sqrt{\det g} dxdy = (1+x)^{-1/2}dxdy$. The Friedlander's model uses instead the Laplace operator associated to the Dirichlet form $\int |\nabla_g u|^2 dxdy = \int (|\partial_x u|^2 + (1+x)|\partial_y u|^2) dxdy$ and is self adjoint with volume form $dxdy$. As a model, the Friedlander operator Δ_g is better than the Laplacian $\Delta_{g,0}$ since it allows explicit computations. Clearly, manifold (Ω_d, g) is a strictly convex domain: In fact, on the geodesic flow starting at $x=0, y=y_0, \xi_0^2 + \eta_0^2 = 1, \xi_0 \in]0, 1[$, one has $x(s) = 2s\xi_0 - s^2\eta_0^2$. Moreover, (Ω_2, g) may be seen as a simplified model for the disk $D(0, 1)$ with polar coordinates (r, θ) , where $r = 1 - x/2$ and $\theta = y$. Multiply reflected light rays become periodic curves in the y variable, as illustrated on Figure 1.1.

Remark 1.2. We will always work with the Dirichlet boundary condition. The Neumann boundary condition can be handled exactly in the same way, providing the same results: one simply modifies the reflexion coefficient in our parametrix construction and replaces zeros of the Airy function Ai by zeros of its derivative Ai' .

We are now in a position to state our main result.

Theorem 1.3. *Let $d \geq 2$. There exists $C > 0, T_0 > 0$ such that for every $a \in]0, 1]$, $h \in (0, 1]$ and $t \in (0, T_0]$, the solution u_a to (1.1) with data $(u_0, u_1) = (\delta_a, 0)$ where δ_a is the Dirac mass at point $(a, 0, \dots, 0) \in \Omega_d$, satisfies*

$$(1.5) \quad |\psi(-h^2 \Delta_g)u_a(t, x)| \leq Ch^{-d} \min(1, (h/t)^{\frac{d-2}{2} + \frac{1}{4}}).$$

The dispersion estimate (1.5) may be compared to (1.4): we notice a $1/4$ loss in the h/t exponent, which we may informally relate to the presence of caustics in arbitrarily small times if a is small. Such caustics occur because optical rays are no longer diverging from

each other in the normal direction, where less dispersion occurs as compared to the \mathbb{R}^d case. We will prove in fact a slightly better estimate than (1.5): the $(h/t)^{1/4}$ factor may be replaced by $h^{1/4} + (h/t)^{1/3}$ for $a \leq h^{1/2}$ (proposition 3.4) and by $(h/t)^{1/2} + a^{1/8}h^{1/4}$ for $a \geq h^{4/7-\varepsilon}$ (Theorem 2.1). In fact, we can track the caustics and therefore our estimate is optimal for $a \geq h^{4/7-\varepsilon}$.

Theorem 1.4. *Let $d \geq 2$ and u_a be the solution to (1.1) with data $(u_0, u_1) = (\delta_a, 0)$. Let $h \in (0, 1]$ and $a \geq h^{4/7-\varepsilon}$. There exists a finite sequence $(t_n)_n$, $1 \leq n \leq \min(a^{-1/2}, a^{1/2}h^{-1/3})$ such that*

$$(1.6) \quad h^{-d}(h/t_n)^{\frac{d-2}{2}}n^{-1/4}a^{\frac{1}{8}}h^{1/4} \lesssim |\psi(-h^2\Delta_g)u_a(t_n, a)| \text{ and } t_n \sim 4n\sqrt{a},$$

As a byproduct, we get that even for $t \in]0, T_0]$ with T_0 small, the $1/4$ loss is unavoidable for a small enough. We will see soon that this optimal loss is due to swallowtail type singularities in the wave front set of u_a .

Remark 1.5. It follows from our proof that theorem 1.3 and 1.4 holds true if one replace $\psi(-h^2\Delta_g)u_a(t, x)$ by $\psi(hD_t)e^{\pm it\sqrt{-\Delta_g}}\delta_{x=a, y=0}$ with $\psi \in C_0^\infty(\mathbb{R}^*)$.

As a consequence of (1.5) and classical arguments, we obtain the following set of Strichartz estimates.

Theorem 1.6. *Let u be a solution of (1.1) on the model domain Ω_d , $d \geq 2$. Then there exists T such that*

$$(1.7) \quad \|u\|_{L^q(0,T)L^r(\Omega)} \leq C_T \left(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)} \right),$$

for (d, q, r) satisfying

$$\frac{1}{q} \leq \left(\frac{d-2}{2} + \frac{1}{4} \right) \left(\frac{1}{2} - \frac{1}{r} \right),$$

and β is dictated by scaling.

In dimension $d = 2$ the known range of admissible indices for which sharp Strichartz hold is in fact slightly larger, see [2]. However, in larger dimensions $d \geq 3$, Theorem 1.3 improves the range of indices for which sharp Strichartz do hold, and does so in a uniform way with respect to dimension, in contrast to [2]. On the other hand, our results are, for now, restricted to a model case of strictly convex domain, while [2] applies to any domain. One may use the model case analysis to extend estimates to any smooth strictly convex domain, as in the counterexample situation [5]. This issue will be addressed elsewhere.

Remark 1.7. One conjectures that the loss in Strichartz estimates in [6] are optimal. This would heuristically match a $1/6$ loss in the dispersion estimate. We plan to address this issue in future work, by proving that the worst time-space points (t_n, a) may be suitably averaged over.

One may then make good use of such Strichartz estimates for the local (and global) Cauchy theory of nonlinear wave equations. We provide one simple example.

Theorem 1.8. *The energy critical wave equation $\square_g u + |u|^{\frac{4}{d-2}}u = 0$ with data $(u_0, u_1) \in H_0^1(\Omega_d) \times L^2(\Omega_d)$ has unique global in time solutions for $3 \leq d \leq 6$.*

1.1. Light propagation, heuristics and degenerate oscillatory integrals. In [9] the second author sketched the main steps of a proof of (1.5) and gave a full description of the geometry of the wave front set. In this work, we provide a complete construction of a suitable parametrix for the wave equation, which we then utilize to obtain decay estimates by (degenerate) stationary phases.

Recall that, at time $t > 0$, one expects the wave propagating from the source of light to be highly concentrated around the sphere of radius t . For a variable coefficients metric, one can make good of this heuristic as long as two different light rays emanating from the source do not cross: in other words, as long as t is smaller than the injectivity radius. One may then construct parametrices using oscillatory integrals, where the phase encodes the geometry of the wave front.

In our situation, the geometry of the wave front becomes singular in arbitrarily small times, depending on the frequency and the distance to the boundary of the source. In fact, a caustic appears right between the first and the second reflexion of the wave front, as illustrated on Figure 1.3 and Figure 1.4 (which is a zoomed version at the relevant time scale). Therefore, we are to investigate concentration phenomena (“caustics”) that may occur near the boundary. Geometrically, caustics are defined as envelopes of light rays coming from our source of light. Each ray is tangent to the caustic at a given point. If one assigns a direction on the caustic, it induces a direction on each ray. Each point outside the caustic (and in the sunny side of the caustic) lies on a ray which has left the caustic and also lies on a ray approaching the caustic. Each curve of constant phase has a cusp where it meets the caustic.

At the caustic point we expect light to be singularly intense. Analytically, caustics can be characterized as points where usual bounds on oscillatory integrals are no longer valid. Oscillatory integrals with caustics have enjoyed much attention: their asymptotic behavior is known to be driven by the number and the order of those of their critical points which are real. Let us consider an oscillatory integral

$$(1.8) \quad u_h(z) = \frac{1}{(2\pi h)^{1/2}} \int_{\zeta} e^{\frac{i}{h}\Phi(z,\zeta)} g(z, \zeta, h) d\zeta, \quad z \in \mathbb{R}^d, \quad \zeta \in \mathbb{R}, \quad h \in (0, 1].$$

We assume that Φ is smooth and that $g(\cdot, h)$ is compactly supported in z and in ζ . If there are no critical points of the map $\zeta \rightarrow \Phi(z, \zeta)$, so that $\partial_{\zeta}\Phi \neq 0$ everywhere in an open neighborhood of the support of $g(\cdot, h)$, then repeated integration by parts (e.g. non stationary phase) yields that $|u_h(z)| = O(h^N)$, for any $N > 0$.

If there are non-degenerate critical points, where $\partial_{\zeta}\Phi = 0$ but $\det(\partial_{\zeta_j\zeta_k}^2\Phi) \neq 0$, then the method of stationary phase applies and yields $\|u_h(z)\|_{L^\infty} = O(1)$. The corresponding canonical form is a Gaussian phase.

If there are degenerate critical points, we define them to be caustics, as $\|u_h(z)\|_{L^\infty}$ is no longer uniformly bounded. The order of a caustic κ is defined as the infimum of κ' so that $\|u_h(z)\|_{L^\infty} = O(h^{-\kappa'})$.

The most simple degenerate phase beyond the Gaussian is $\Phi_F(z, \zeta) = \frac{\zeta^3}{3} + z_1\zeta + z_2$, which corresponds to a fold with order $\kappa = \frac{1}{6}$. A typical example is the Airy function. The caustic

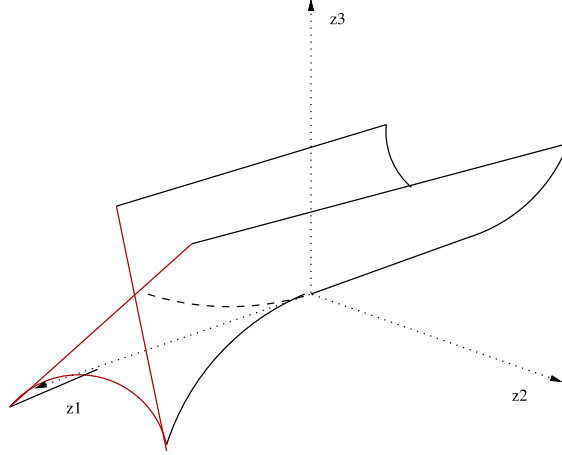


FIGURE 1.2. The caustic for the swallowtail catastrophe.

is given by $z_1 = 0$ and the illuminated side is $z_1 < 0$. The next canonical form is given by a phase function which is a polynomial of degree 4, namely $\Phi_C(z, \zeta) = \frac{\zeta^4}{4} + z_1 \frac{\zeta^2}{2} + z_2 \zeta + z_3$ whose order is $\kappa = \frac{1}{4}$; its associated integral is called Pearcey's function and it produces a cusp singularity on the caustic which is parametrized by $z_1 = -3s^2, z_2 = 2s^3$.

Finally, we conclude this brief overview with the swallowtail integral (which is an oscillatory integral with four coalescing saddle points) whose canonical form is given by a polynomial of degree 5, $\Phi_S(z, \zeta) = \frac{\zeta^5}{5} + z_1 \frac{\zeta^3}{3} + z_2 \frac{\zeta^2}{2} + z_3 \zeta + z_4$: the caustic surface of the swallowtail is defined by the condition that two or more *real* saddle points are equal: it is pictured on Figure 1.2. In the event that two simple saddle points undergo confluence when $z \rightarrow z_0$, then the uniform asymptotic behavior of (1.8) contains terms involving the Airy function and its derivatives multiplied by powers of $h^{-\frac{1}{2} + \frac{1}{3}}$; the caustic surface is smooth ($z_1 < 0$ on the figure). If three simple saddles coalesce as $z \rightarrow z_0$, then the uniform asymptotic behavior of (1.8) can be described by terms containing the Pearcey function and its first-order derivatives, each multiplied by a power of $h^{-\frac{1}{2} + \frac{1}{4}}$; the caustic surface has cusps (two of them in the $z_1 > 0$ region on the figure). The swallowtail enters in picture when four simple saddle points of (1.8) undergo confluence as $z \rightarrow z_0$ (which is $z_0 = 0$ on the figure).

Such integrals will play a crucial role in the proof of Theorem 1.3. Between two consecutive reflections of the wave propagating along the boundary, we shall construct a parametrix of the form

$$u(z, h) = \frac{1}{h^2} \int_{\mathbb{R}^2} e^{\frac{i}{h} \Phi(z, \zeta, \eta)} g(z, \eta, \zeta, h) d\zeta d\eta,$$

where the phase is essentially $\Phi(z, \zeta, \eta) \approx \eta \Phi_C(z, \zeta)$, with $z = (t, x, y)$, η/h is the Fourier variable associated to the tangential variable y and $\zeta = \xi/\eta$ where ξ/h is the Fourier variable associated to the normal variable x . Note we may restrict to $\eta \in (1/2, 2)$ which corresponds precisely to waves propagating along the boundary and explains the (η, ζ)

parametrization for the oscillatory integral. For a particular value z_S of $z = (t, x, y)$, this phase will have a saddle point of order 4; it corresponds to $\partial_\eta \Phi = 0 = \Phi_C(z_S, \zeta)$ and $\partial_\zeta \Phi_C = \partial_\zeta^2 \Phi_C = \partial_\zeta^3 \Phi_C = 0$: the geometric picture is that of a swallowtail singularity, but the decay loss is that of the Pearcey's integral, i.e. $h^{1/4}$. For $z \neq z_S$, our oscillatory integral will have only critical points of order at most 3, corresponding e.g. to $\partial_\eta \Phi = 0 = \Phi_C(z_S, \zeta)$ and $\partial_\zeta \Phi_C = \partial_\zeta^2 \Phi_C = 0$: the picture is, at worst, that of cusps and the loss is that of the Airy function, i.e. $h^{1/6}$. Finally, we notice Figures 1.2 and 1.4 picture the same singularity formation: in 1.2, up to translations, $z_1 = t$, $z_2 = -x$ and $z_3 = y$; $z_1 < 0$ corresponds to the (smooth) refocusing wave front in the left part of 1.4 while two cusps form on the right part after the swallowtail singularity.

1.2. An outline of the proof. Let us mention the main ideas of the proof of Theorem 1.3. First, we may reduce to the two dimensional case, as the tangential directions will produce the usual decay factor when we integrate them out, see Section 4.

Let $h \in (0, 1]$ be a small parameter ($1/h$ will later be the spectral frequency) and $0 < a \ll 1$ the distance of the source to the boundary. We assume a to be small as we are interested in highly reflected waves, which we do not observe if the waves do not have time to reach the boundary.

From the spectral analysis which will be recalled in Section 3.1, we have an explicit representation for the Green function associated to the half-wave initial value problem with a Dirac at (a, b) as initial condition at time s :

$$(1.9) \quad G((x, y, t), (a, b, s)) = \sum_{k \geq 1} \int_{\mathbb{R}} e^{\pm i(t-s)\sqrt{\lambda_k(\eta)}} e^{i(y-b)\eta} e_k(x, \eta) e_k(a, \eta) d\eta$$

where $\lambda_k(\eta) = \eta^2 + \eta^{4/3}\omega_k$, with $-\omega_k$ a zero of the Airy function and the $e_k(x, \eta)$ are explicit, real-valued functions which are defined in Section 3.1. We now record several remarks that will be of help later and relate to various phase space localizations.

Remark 1.9. We may perform a spectral localization at $\lambda_k(\eta) \sim h^{-1}$, which corresponds to inserting a smooth, compactly supported away from zero $\psi_2(h\sqrt{\lambda_k(\eta)})$; on the flow, this is nothing but $\psi_2(hD_t)$ and this smoothes out the Green function. Then we are dealing with a semi-classical boundary value problem with small parameter h . With the usual notations $\tau = \frac{h}{i}\partial_t$, $\eta = \frac{h}{i}\partial_y$, $\xi = \frac{h}{i}\partial_x$, the characteristic set of our operator is given by

$$\tau^2 = \xi^2 + (1+x)\eta^2$$

The hyperbolic (resp. elliptic) subset of the cotangent bundle of the boundary $x = 0$ is $|\tau| > |\eta|$, (resp. $|\tau| < |\eta|$) and the gliding subset is $|\tau| = |\eta|$. From $\tau^2 = (hD_t)^2 = h^2\lambda_k(D_y)$, one gets at the symbolic level on the micro-support of any gallery mode associated to ω_k

$$(1.10) \quad \eta^{4/3}h^{2/3}\omega_k = \xi^2 + x\eta^2.$$

Remark 1.10. We may also localize with $\psi_1(hD_y)$, with $\psi_1 \in C_0^\infty([0, \infty[)$, which correspond to a Fourier localization along the tangential (e.g. y) direction (notice such a truncation is easily seen to commute with the equation, hence the flow). Since we are not interested

with waves transverse to the boundary, we may and will assume that on the support of $\psi_1(\eta)\psi_2(h\sqrt{\lambda_k(\eta)})$ one has $k \leq \varepsilon h^{-1}$ with ε small. This is compatible with (1.10) since $\omega_k \simeq k^{2/3}$ and $k \leq \varepsilon h^{-1}$ is equivalent to $|\xi| \lesssim \varepsilon^{2/3}$. This fact will later have its importance when $a \leq h^{1/2}$.

Remark 1.11. Irrespective of the position of a relative to h , the remaining part of the Green function, will be essentially transverse and see at most one reflexion for $t \in [0, T_0]$, with T_0 small (depending on the above choice of ε). Hence, it can be dealt with as in [1] to get the free space decay and we will ignore it in the upcoming analysis.

Remark 1.12. Finally, the symmetry of G (or its suitable spectral truncations) with respect to x and a will be of great importance: it allows us to restrict the computation of the L^∞ norm to the region $0 \leq x \leq a$.

Now, we consider initial data $u_0(x, y) = \psi_2(h\sqrt{-\Delta_g})\psi_1(hD_y)\delta_{x=a, y=0}$ where the ψ_j are those of remark 1.10. We will use different arguments depending on the respective position of a and h .

The first case is $a \gg h^{4/7}$: there, we follow ideas of [6] and write a parametrix for the wave equation as a superposition of localized waves for which we can compute the wave front set and hence the singularities that appear at different times and locations. The construction of [6] has to be significantly altered to allow for the range $h^{4/7} \ll a \leq h^{1/2}$, with a phase which is less explicit but prevents amplifying factors at each reflexion that induced the $a > h^{1/2}$ restriction in [6].

The second case corresponds to data for which the distance a to the boundary is such that $0 < a \lesssim h^{1/2}$: we write the contribution of our data which is localized in a $h^{1/4}$ cone of tangential directions as the $L^2(\Omega)$ orthogonal sum of whispering gallery modes and prove that after a time t the corresponding wave remains frequency localized in the same cone of directions of size $h^{1/4}$, at least up to smooth remainders. While not quite as strong as a microlocal propagation of singularities result, this allows for the use of Sobolev embedding theorem to recover the “dispersion” by using the size of the Fourier support. The contribution of data corresponding to directions with angles with the boundary greater than $h^{1/4}$ may be dealt with separately, using a crude parametrix construction, as they involve only cusp-type singularities.

Notice that there is an overlap between the two regions: in fact the parametrix construction obviously provides better bounds in the overlap region, both in size (we gain an $a^{1/8}$ factor in the worst case) and position (the swallowtail occurs exactly once in between two consecutive reflexions). Had we reproduced the parametrix construction from [6], we would have an epsilon loss in the dispersion estimate because of the $a \sim h^{1/2}$ region. We thought it was of independent interest to quantify how “far” below $h^{1/2}$ the construction could be pushed while retaining the most interesting features of [6].

Remark 1.13. Figure 1.3 illustrates the propagation of (part of) the wavefront set of the Dirac data; the second picture is a zoomed version of the first one and shows in detail the formation of the swallowtail singularity for the part of the wave front moving along directions which are initially tangent to the boundary.

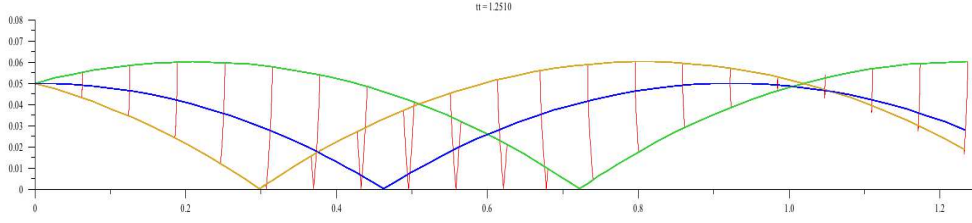


FIGURE 1.3. Propagation of the wavefront

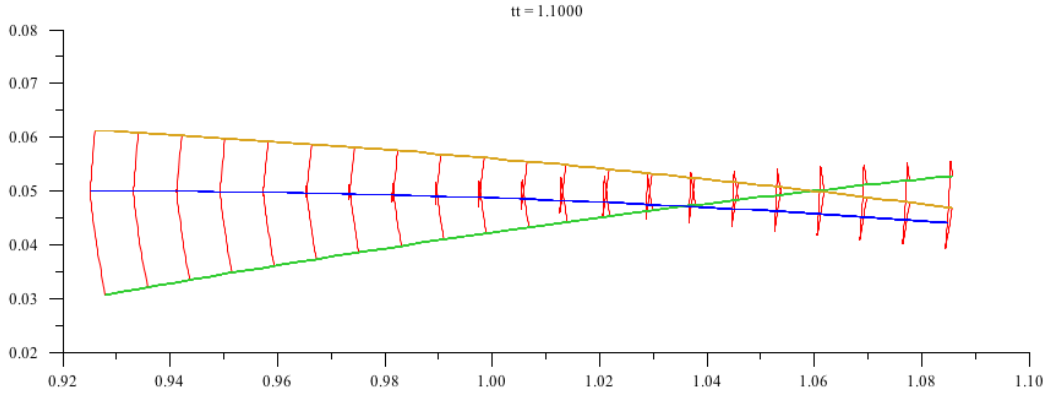


FIGURE 1.4. The formation of a swallowtail singularity just after the first reflection (zoomed image)

Finally, Theorem 1.3 is obtained for $a \geq h^{4/7-\varepsilon}$ in Section 2, Theorem 2.1, and for $a \leq h^{1/2}$ in Section 3, Proposition 3.4. Theorem 1.4 is obtained in section 2, as a remark at the end of the proof of Proposition 2.14.

2. PARAMETRIX FOR $a > h^{4/7}$

This section is devoted to the construction, modulo $O(h^\infty)$, of the Green function in the case $a \geq h^{4/7-\varepsilon}$. The Green function is represented in proposition 2.4 as a superposition of $O(a^{-1/2})$ reflected waves. We give a precise analysis of the Lagrangian in the phase space associated to each reflected wave. This geometric analysis allows us to track the degeneracy of the phases when we apply phase stationary arguments. Our main dispersive estimate will be Theorem 2.1.

Let us set $\hbar = h/\eta$ and $\mathbb{P} = (-i\hbar\partial_x)^2 + 1 + x - (-i\hbar\partial_t)^2$. For $a \geq 0$, we denote by $\Lambda_a \subset T^*\mathbb{R}$ the Lagrangian

$$(2.1) \quad \Lambda_a = \{(t', \tau') \mid \exists \theta \in \mathbb{R} \text{ s.t. } t' = -2\theta\sqrt{1+a+\theta^2}, \tau' = \sqrt{1+a+\theta^2}\}.$$

The set Λ_a may be parametrized by t' . Let $\psi_a(t')$ be the unique function such that $\psi_a(0) = 0$ and $\Lambda_a = \{(t', \psi'_a(t'))\}$. Let us set $\rho = 1 + a$ and $\theta = \sqrt{\rho}z$, then $(t')^2 = 4\rho^2(z^2 + z^4)$, from

which we get

$$2z^2 + 1 = \sqrt{1 + (t')^2/\rho^2} \implies \psi_a'^2 = \rho(1 + z^2) = \frac{\rho}{2}(1 + \sqrt{1 + (t')^2/\rho^2})$$

and as $\psi_a' = \tau' > 0$,

$$\psi_a' = \sqrt{\rho}(1 + t'^2/(8\rho^2) + O(t'^4));$$

finally, by integration, as $\psi_a(0) = 0$,

$$(2.2) \quad \psi_a(t') = \sqrt{\rho}(t' + \frac{t'^3}{24\rho^2} + O(t'^5))$$

2.1. A singular integral representation for the data. We start by a suitable decomposition of the smoothed Dirac as an inverse Fourier transform of a superposition of Airy functions.

Lemma 2.1. *Let $\chi_1 \in C_0^\infty((-\theta_0, \theta_0))$ with small θ_0 . There exists a symbol $\sigma_0(t', \hbar)$ of degree 0 with an asymptotic expansion in \hbar , supported in a neighborhood of $t' = 0$ with the following properties: let*

$$(2.3) \quad \tilde{u}_0(t, x, h, \hbar) = \frac{1}{2\pi\hbar} \int e^{\frac{i}{\hbar}(\zeta(t-t') + s(x+1-\zeta^2) + \frac{s^3}{3})} e^{\frac{i}{\hbar}\psi_a(t')} \sigma_0(t', \hbar) \frac{dt' ds d\zeta}{(2\pi\hbar)^2}$$

then \tilde{u}_0 is such that, for $x > -1$,

(1) *The wave front set of \tilde{u}_0 is included in $\tau > 0$. In fact,*

$$WF_h(\tilde{u}_0) \subset \{\tau \in [\sqrt{1+a}, \tau_0]\},$$

where τ_0 is related only to the size of the support of σ_0 in t' . Moreover,

$$\mathbb{P}\tilde{u}_0 = 0.$$

(2) *The initial data $\tilde{u}_0(0, x, h, \hbar)$ is a smoothed out Dirac, that is*

$$\tilde{u}_0(0, x, h, \hbar) = \frac{1}{(2\pi\hbar)^2} \int e^{\frac{i}{\hbar}(x-a)\theta} \chi_1(\theta) d\theta + O_{C^\infty}(\hbar^\infty).$$

Proof. Consider the time Fourier transform of \tilde{u}_0 ,

$$\begin{aligned} \hat{\tilde{u}}_0(\tau/\hbar, x, h, \hbar) &= \int e^{-it\tau/\hbar} \tilde{u}_0(t, x, h, \hbar) dt \\ &= \hbar^{\frac{1}{3}} \int \text{Ai}(\hbar^{-\frac{2}{3}}(x+1-\tau^2)) e^{\frac{i}{\hbar}(\psi_a(t') - \tau t')} \sigma_0(t', \hbar) \frac{dt'}{(2\pi\hbar)^2}. \end{aligned}$$

Therefore, $\hat{\tilde{u}}_0$ is an average (with compact support in t') of solutions to the equation

$$(\frac{\hbar}{i}\partial_x)^2 f + (1+x-\tau^2)f = 0.$$

From $\partial_{t'}(\psi_a(t') - \tau t') = 0$, we get $\tau = \psi_a'(t')$ and therefore there exists θ such that $\tau = \sqrt{1+a+\theta^2}$, which proves the claim on $WF_h(\tilde{u}_0)$.

We proceed with the second part of the statement, regarding the initial data,

$$\tilde{u}_0(0, x, h, \hbar) = \frac{1}{(2\pi\hbar)^2 2\pi\hbar} \int e^{\frac{i}{\hbar}(s(x+1-\zeta^2)+s^3/3-t'\zeta+\psi_a(t'))} \sigma_0 dt' ds d\zeta.$$

Let $\phi(t', x, s, \zeta) = s(x+1-\zeta^2) + s^3/3 - t'\zeta$ and denote by \mathcal{C}_ϕ the set

$$\mathcal{C}_\phi = \{(t', x, s, \zeta) \text{ s.t. } \partial_s \phi = \partial_\zeta \phi = 0\}.$$

The equations defining \mathcal{C}_ϕ read $x+1+s^2=\zeta^2$ and $2s\zeta+t'=0$. From the first equation, we get $\zeta \neq 0$ on \mathcal{C}_ϕ (recall $x > -1$). Now,

$$\text{Hess}_{s,\zeta} \phi = \begin{pmatrix} 2s & -2\zeta \\ -2\zeta & -2s \end{pmatrix},$$

and $\det(\text{Hess}_{s,\zeta} \phi) \neq 0$ on \mathcal{C}_ϕ . Therefore \mathcal{C}_ϕ is a smooth manifold.

Denote by π_1 the projection from \mathcal{C}_ϕ to $T^*\mathbb{R}_x$, that is

$$\pi_1((t', x, s, \zeta) \in \mathcal{C}_\phi) = (x, \partial_x \phi) = (x, s).$$

and by π_2 the projection from \mathcal{C}_ϕ to $T^*\mathbb{R}_{t'}$, that is

$$\pi_2((t', x, s, \zeta) \in \mathcal{C}_\phi) = (t', -\partial_{t'} \phi) = (t', \zeta),$$

For $\tau' \neq 0$, we have

$$\pi_2^{-1}(t', \tau') = (t', x = -1 + \tau'^2 - t'^2/(4\zeta^2), s = -t'/2\zeta, \zeta = \tau').$$

Therefore \mathcal{C}_ϕ induces a canonical transformation from $T^*\mathbb{R}_{t'} \setminus \{\tau' = 0\}$ to $T^*\mathbb{R}_x$ defined by

$$\chi(t', \tau') = (x = -1 + \tau'^2 - t'^2/4\tau'^2, \xi = -t'/2\tau').$$

Notice that

$$\chi(\Lambda_a) = (x = a, \xi = \theta) = T_{x=a}^*,$$

and χ is a symplectic isomorphism from a neighborhood of $(t', \tau') = (0, \sqrt{1+a})$ onto a neighborhood of $(x, \xi) = (a, 0)$.

The remaining part of the argument is standard: denote by $G(t', x) = \phi(t', x', s_c, \zeta_c)$ where $(s_c, \zeta_c > 0)$ is the unique solution of $1+x+s_c^2=\zeta_c^2$ and $t'+2s_c\zeta_c=0$, then $G(0, a) = 0$, as $s_c(0, a) = 0$ and $\zeta_c(0, a) = \sqrt{1+a}$. By stationary phase in (s, ζ) we get

$$\tilde{u}_0(0, x, h, \hbar) = \frac{1}{(2\pi\hbar)^2} \int e^{\frac{i}{\hbar}(G(t', x) + \psi_a(t'))} A_0(t', x, \hbar) \sigma_0(t', \hbar) dt'$$

where $A_0(t', x, \hbar)$ is an elliptic symbol of order 0. From $\partial_{t'} G(t', a) + \psi'_a(t') = 0$ and $G(0, a) = 0$, we get $G(t', a) = -\psi_a(t')$, and therefore

$$G(t', x) + \psi_a(t') = (x-a)H_a(t', x) \text{ with } \partial_{t'} H_a(0, a) \neq 0,$$

and by change of variables $\Theta = H_a(t', x)$ and using that for all F there exists G such that

$$\int e^{\frac{i}{\hbar}(x-a)\Theta} F(\Theta, x, \hbar) d\Theta = \int e^{\frac{i}{\hbar}(x-a)\Theta} G(\Theta, \hbar) d\Theta + O(\hbar^\infty)$$

we obtain the desired conclusion, since by the above canonical transformation the map $\sigma_0(t', h) \mapsto G(\Theta, \hbar)$ is elliptic of degree 0.

□

Set $g_0(t', \hbar) = e^{\frac{i}{\hbar}\psi_a(t')}\sigma_0(t', \hbar)$. We proceed with

Lemma 2.2. *Let $c > 0$, $\varepsilon > 0$, then, with $\rho = 1 + a$,*

$$(2.4) \quad \sup_{\tau \leq \sqrt{\rho} - c\hbar^{2/3-\varepsilon}} |\hat{g}_0(\tau/\hbar, \hbar)| \in O(\hbar^\infty).$$

Proof. Notice that $\hat{g}_0(\tau/\hbar)$ behaves like an Airy function from the geometry of Λ_a so the estimate on \hat{g}_0 is really the classical estimate on Ai . We provide a direct argument: for $\tau < \sqrt{\rho}$,

$$\hat{g}_0(\tau/\hbar, \hbar) = \int e^{\frac{i}{\hbar}(\psi_a(t) - t\tau)} \sigma_0(t, \hbar) dt,$$

and we may integrate by parts using $L = (\psi'_a(t) - \tau)^{-1} \frac{\hbar}{i} \partial_t$ (recall $\psi'_a(t) = \sqrt{\rho}(1 + t^2/8\rho^2 + O(t^4))$). Notice that

$$({}^tL)^N(\sigma_0) = \hbar^N \sum_{j=0}^N \frac{\alpha_{j,N}(t)}{(\psi'_a(t) - \tau)^{2N-j}}$$

where $\alpha_{j,N}(t) = t^{\lfloor N-2j \rfloor_+} \beta_{j,N}(t)$ (by induction, as $\psi''_a(t) = O(t)$ and $\lfloor N-2j \rfloor_+ + 1 \geq \lfloor N-2j+1 \rfloor_+$ as well as $\lfloor \lfloor N-2(j-1) \rfloor_+ - 1 \rfloor_+ \geq \lfloor N-2j+1 \rfloor_+$). As such, it remains to check that for $t \in [-1, +1]$ and $\alpha \in]0, 1]$

$$\hbar^N \frac{t^{\lfloor N-2j \rfloor_+}}{(\alpha + t^2)^{2N-j}} \leq C_{N,j} \frac{\hbar^N}{\alpha^{3N/2}},$$

which is trivial if $j \geq N/2$ and follows from setting $t = \sqrt{\alpha}s$ if $j < N/2$. □

Remark 2.3. One may also prove that there exists $\tau_0 > 0$ (related to the support of σ_0) such that

$$(2.5) \quad \sup_{\tau \geq \tau_0} |\hat{g}_0(\tau/\hbar, \hbar)| \in O(\hbar^\infty).$$

2.2. Digression on Airy functions. We recall a few well-known facts about Airy functions: let $z > 0$,

$$\text{Ai}(-z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(s^3/3 - sz)} ds = A_+(z) + A_-(z),$$

with $A_+ = \bar{A}_-$ and

$$A_-(z) = \frac{1}{2\pi} \int_0^{+\infty} e^{i(s^3/3 - sz)} ds = \frac{1}{2\sqrt{\pi}z^{1/4}} e^{i\pi/4} e^{-\frac{2}{3}iz^{3/2}} \exp \Upsilon(z^{3/2}) = \frac{1}{z^{1/4}} e^{i\pi/4} e^{-\frac{2}{3}iz^{3/2}} \Psi_-(z)$$

with $\exp \Upsilon(z^{3/2}) \sim (1 + \sum_{l \geq 1} c_l z^{-\frac{3l}{2}}) \sim 2\sqrt{\pi} \Psi_-(z)$ as $z \rightarrow +\infty$. We also define $\Psi_+(z) = \bar{\Psi}_-(\bar{z})$. Therefore, we have

$$\frac{A_-(z)}{A_+(z)} = i e^{-\frac{4}{3}iz^{3/2}} e^{iB(z^{3/2})} \text{ with } iB = \Upsilon - \bar{\Upsilon}.$$

Notice that for $u \in \mathbb{R}_+$, $B(u) \in \mathbb{R}$ and $B(u) \sim \sum_{j \geq 1} b_j u^{-j}$ for $u \rightarrow +\infty$.

2.3. The parametrix construction. Let $F(\zeta, \hbar)$ be a function with compact support in $\zeta \in [1 + ch^{\frac{2}{3}-\varepsilon}, \zeta_0]$. Define

$$u(t, x, \hbar) = \frac{1}{2\pi\hbar} \int e^{\frac{i}{\hbar}(\zeta t + s(x+1-\zeta^2) + s^3/3)} F(\zeta, \hbar) d\zeta.$$

One easily checks that $\mathbb{P}u = 0$ and the trace on $x = 0$ is

$$u(t, 0, \hbar) = \hbar^{-\frac{2}{3}} \int e^{\frac{i}{\hbar}t\zeta} (A_+ + A_-)(\hbar^{-2/3}(\zeta^2 - 1)) F(\zeta, \hbar) d\zeta.$$

Define f by $F(\zeta, \hbar) = \int \exp(-it'\zeta/\hbar) f(t', \hbar) dt'$, then

$$u(t, 0, \hbar) = J_+(f) + J_-(f),$$

where J_{\pm} corresponds to canonical transformations j_{\pm} on $T^*\mathbb{R}_t \cap \{\tau \geq 1\}$,

$$j_{\pm}(t', \tau') = (t = t' \mp 2\tau\sqrt{\tau^2 - 1}, \tau = \tau').$$

We now set up a few notations:

- a cut-off function $\chi_0(\eta) \in C_0^\infty((1/2, 5/2))$ such that $\chi_0 = 1$ on $[1, 2]$;
- recall $\chi_1 \in C_0^\infty((-\theta_0, \theta_0))$ with small θ_0 ;
- let $a \in [h^{\frac{2}{3}-\varepsilon}, a_0]$, with a_0 small;
- let $\beta > 0$ be such that $\sqrt{1+a} - \sqrt{1+a\beta} \geq ca$, for all $a \in [0, a_0]$;
- let $\chi_2(u) \in C^\infty$ with $\chi_2(u) = 0$ for $u \leq \beta/2$ and $\chi_2(u) = 1$ for $u \geq \beta$;
- let $\chi_3(\zeta) \in C^\infty$ with $\chi_3(\zeta) = 1$ for $3/4 \leq \zeta \leq \zeta_0$ and $\chi_3(\zeta) = 0$ for $\zeta \geq \zeta_1$ or $\zeta \leq 1/2$ (with $\zeta_1 > \zeta_0$, $\zeta_0 - 1 > 0$ and small and $\zeta_1 - 1$ small).

Define

$$\begin{aligned} v_N(t, x, y, h) &= \frac{1}{(2\pi\hbar)^2} \int e^{i\frac{\eta}{\hbar}y} u_N(t, x, h/\eta) \eta \chi_0(\eta) d\eta, \\ u_N(t, x, \hbar) &= \frac{(-i)^N}{2\pi\hbar} \int e^{\frac{i}{\hbar}(t\zeta + s(x+1-\zeta^2) + s^3/3 - 4N/3(\zeta^2-1)^{\frac{3}{2}} + \hbar NB((\zeta^2-1)^{\frac{3}{2}}/\hbar))} \\ &\quad \times \chi_2((\zeta^2 - 1)/a) \chi_3(\zeta) \hat{g}_0(\zeta/\hbar, \hbar) ds d\zeta, \\ v(t, x, y, h) &= \sum_{0 \leq N \leq C_0/\sqrt{a}} v_N(t, x, y, h). \end{aligned}$$

and let $P = \partial_t^2 - (\partial_x^2 + (1+x)\partial_y^2)$.

Proposition 2.4. *There exists C_0 such that the following holds true:*

- (1) v is a solution to $Pv = 0$ for $x > -1$;
- (2) its trace on the boundary, $v(t \in [0, 1], x = 0)$ is $O_{C^\infty}(h^\infty)$;
- (3) at time $t = 0$, we have

$$v(0, x, y, h) - (2\pi\hbar)^{-2} \int e^{i(\eta y + (x-a)\xi)} \chi_0(\eta) \chi_1(\xi/\eta) d\eta d\xi = O_{C^\infty}(h^\infty).$$

Remark 2.5. Here and thereafter, $f(z, h) \in O_{C^\infty}(h^\infty)$ for $z \in \Gamma$ if, uniformly in $a \in [h^{\frac{2}{3}-\varepsilon}, 1]$,

$$\forall \alpha, N, \exists C_{\alpha, N} \text{ s.t. } \sup_{z \in \Gamma} |\partial_z^\alpha f(z, h)| \leq C_{\alpha, N} h^N.$$

Proof. Obviously v is defined by a finite sum and each v_N is a solution to $Pv_N = 0$. We postpone the rest of the proof to subsection 2.5. \square

Remark 2.6. We may also define v by a sum from $-C_0/\sqrt{a}$ to C_0/\sqrt{a} , and replace $t \in [0, 1]$ by $t \in [-1, 1]$. The equation enjoys time symmetry and therefore the two points of view are equivalent.

We start by studying u_N ; from there, we may obtain information on v_N by integration over η . This, however, is a non trivial matter, as $\hbar = h/\eta$ and integration over η has an effect on $\exp(iNB((\zeta^2 - 1)^{\frac{3}{2}}/\hbar))$.

Let

$\phi_{a, N, \hbar}(t, x, t', s, \zeta) = (t - t')\zeta + s(x + 1 - \zeta^2) + s^3/3 - 4N/3(\zeta^2 - 1)^{\frac{3}{2}} + \hbar NB((\zeta^2 - 1)^{\frac{3}{2}}/\hbar) + \psi_a(t')$,
so that

$$u_N(t, x, \hbar) = \frac{(-i)^N}{2\pi\hbar} \int e^{i\phi_{a, N, \hbar}} \chi_2((\zeta^2 - 1)/a) \chi_3(\zeta) \sigma_0(t', \hbar) dt' ds d\zeta.$$

Notice that

- t' takes values in a compact set close to $t' = 0$;
- ζ takes values in a compact set close to $\zeta = 1$;
- the s integral is oscillatory, and as the symbol is independent of s , this yields an Airy function (something we will use only to check the trace condition in Proposition 2.4).

Let us set

$$\mathcal{C}_{a, N, \hbar} = \{(t, x, t', s, \zeta) \text{ s.t. } \partial_{t'} \phi_{a, N, \hbar} = \partial_s \phi_{a, N, \hbar} = \partial_\zeta \phi_{a, N, \hbar} = 0\}$$

we therefore get a system of three equations defining $\mathcal{C}_{a, N, \hbar}$,

$$\begin{aligned} \zeta &= \psi'_a(t') \\ x &= \zeta^2 - 1 - s^2 \\ t &= t' + 2s\zeta + 4N\zeta(\zeta^2 - 1)^{1/2} \left(1 - \frac{3}{4}B'((\zeta^2 - 1)^{\frac{3}{2}}/\hbar)\right). \end{aligned}$$

Notice that on the support of the symbol in the definition of u_N , we have $\zeta \in [\sqrt{1 + a\beta/2}, \zeta_1]$ with $\zeta_1 \sim 1$. We can thus localize further the symbol with $\chi_4(s) \in C_0^\infty$, $\chi_4 = 1$ for $s \in [-\zeta_1, \zeta_1]$, as for $|s| > |\zeta|$, we will have $x = \zeta^2 - 1 - s^2 < -1$ and as such the contribution of $1 - \chi_4$ will be $O_{C^\infty}(h^\infty)$ (by integration by parts in s) in the $x \geq -1$ region.

Remark 2.7. In fact, one may localize closer to $s = 0$: if $\chi_4(s) = 1$ on $[-\sqrt{\zeta_1^2 - 1}, \sqrt{\zeta_1^2 - 1}]$, the same argument provides a remainder term for $x \geq -\varepsilon_0$. Hence, localizing s close to 0 implies ζ_1 close to 1 and therefore θ_0 smaller and smaller and the same for a_0 .

We may parametrize $\mathcal{C}_{a,N,h}$ by (s, θ) when they are close to the origin:

$$\begin{aligned} x &= a + \theta^2 - s^2 \\ t &= 2\sqrt{1+a+\theta^2}(s - \theta + 2N\sqrt{a+\theta^2}(1 - \frac{3}{4}B'((a+\theta^2)^{\frac{3}{2}}/\hbar))) \\ t' &= -2\theta\sqrt{1+a+\theta^2} \\ s &= s \\ \zeta &= \sqrt{1+a+\theta^2}. \end{aligned}$$

Notice that $(s, \theta) \rightarrow (s, t' = -2\theta\sqrt{1+a+\theta^2})$ is a local diffeomorphism in a neighborhood of $(0, 0)$, which ensures that $\mathcal{C}_{a,N,h}$ is a smooth 2D manifold.

Let us denote by $\Lambda_{a,N,h}$ the image of $\mathcal{C}_{a,N,h}$ by the map

$$(t, x, t', s, \zeta) \rightarrow (x, t, \xi = \partial_x \phi_{a,N,h}, \tau = \partial_t \phi_{a,N,h});$$

then $\Lambda_{a,N,h}$ is a Lagrangian submanifold which is parametrized by (s, θ) :

$$\begin{aligned} x &= a + \theta^2 - s^2 \\ t &= 2\sqrt{1+a+\theta^2}(s - \theta + 2N\sqrt{a+\theta^2}(1 - \frac{3}{4}B'((a+\theta^2)^{\frac{3}{2}}/\hbar))) \\ \xi &= s \\ \tau &= \sqrt{1+a+\theta^2}. \end{aligned}$$

Lemma 2.8. *The Lagrangian submanifold $\Lambda_{a,N,h}$ is smooth and its parametrization by (s, θ) is one to one.*

Proof. One has first to verify that at each point (s, θ) , the differential of the map $(s, \theta) \mapsto (x, t, \xi, \tau)$ is injective. But this is obvious since $(\frac{\partial \xi}{\partial s}, \frac{\partial \xi}{\partial \theta}) = (1, 0)$, $\frac{\partial \tau}{\partial \theta} = \theta\tau^{-1/2}$, and if $\theta = 0$, then $\frac{\partial t}{\partial \theta}(s, 0) = -2\sqrt{1+a}$. The map is clearly one to one as $t(s, \theta) \neq t(s, -\theta)$ for $\theta \neq 0$. \square

We digress for a while and explain how to add the y variable. In the definition of $v_N(t, x, y, h)$, we have a phase function

$$\Psi_{a,N,h}(t, x, y, t', s, \zeta, \eta) = \eta y + \eta \phi_{a,N,h/\eta}(t, x, t', s, \zeta),$$

from which we get $\partial_y \Psi_{a,N,h} = \eta$, and

$$\partial_\eta \Psi_{a,N,h} = y + \psi_a(t') + \zeta(t - t') + s(x + 1 - \zeta^2) + s^3/3 + N(\zeta^2 - 1)^{3/2}(-4/3 + B'((\zeta^2 - 1)^{\frac{3}{2}}\eta/h))$$

and the full Lagrangian $\Lambda_{a,N,h} \subset T^*\mathbb{R}^3$ is the set of points $(x, y, t, \xi, \eta, \tau)$ such that there exist (s, θ, η) solution to

$$\begin{aligned} x &= a + \theta^2 - s^2 \\ y &= -\psi_a(-2\theta(1+a+\theta^2)^{\frac{1}{2}}) - 2s(1+a+\theta^2) + \frac{2}{3}s^3 \\ &\quad - N(a+\theta^2)^{\frac{1}{2}}(3+2a+2\theta^2)\left(\frac{4}{3} - B'((\zeta^2-1)^{\frac{3}{2}}\eta/h)\right) \\ t &= 2\sqrt{1+a+\theta^2}(s-\theta+2N(a+\theta^2)^{\frac{1}{2}}(1-\frac{3}{4}B'((\zeta^2-1)^{\frac{3}{2}}\eta/h))) \\ \xi &= \eta s \\ \eta &= \eta \\ \tau &= \eta\sqrt{1+a+\theta^2}. \end{aligned}$$

Remark 2.9. Notice that for $N = 0$, having $t = 0$ in $\Lambda_{a,0,h}$ is equivalent to having $s = \theta$. This implies $x = a$ and then $y = 0$ is a consequence of

$$(2.6) \quad \psi_a(-2\theta(1+a+\theta^2)^{\frac{1}{2}}) = -2\theta(1+a+\theta^2) + \frac{2}{3}\theta^3,$$

and observe that (2.6) holds true since $\psi'_a(t') = (1+a+\theta^2)^{\frac{1}{2}}$ for $t' = -2\theta(1+a+\theta^2)^{\frac{1}{2}}$ and $\psi_a(0) = 0$. Therefore we can explicitly compute $\psi_a(t')$, as $\theta = -t'(1+a+((1+a)^2+t'^2)^{\frac{1}{2}})^{-\frac{1}{2}}/\sqrt{2}$.

2.4. A suitable change of coordinates. We now perform a rescaling of our coordinates that provides some useful reductions.

Set

$$t = \sqrt{a}T, \quad x = aX, \quad y = -t\sqrt{1+a} + a^{\frac{3}{2}}Y,$$

and define $\gamma_N(T, X, Y, h)$, $w_N(T, X, \hbar)$ as

$$\begin{aligned} v_N(t, x, y, h) &= \gamma_N(t/\sqrt{a}, x/a, y + ta^{-3/2}\sqrt{1+a}, h) \\ u_N(t, x, \hbar) &= a^{-\frac{1}{2}}e^{\frac{i}{\hbar}t\sqrt{1+a}}w_N(t/\sqrt{a}, x/a, \hbar). \end{aligned}$$

We define

$$\tilde{\psi}_a(T') = a^{-\frac{3}{2}}(\psi_a(\sqrt{a}T') - \sqrt{a}\sqrt{1+a}T').$$

Notice that $\tilde{\psi}_a$ is C^∞ in (T', a) , with support in $\sqrt{a}|T'| \lesssim 1$ and

$$(2.7) \quad \tilde{\psi}_a(T') = \frac{T'^3}{24\rho^{\frac{3}{2}}}(1 + O(aT'^2)) \quad (\text{recall } \rho = 1+a).$$

Set

$$\zeta^2 - 1 = az, \quad s = \sqrt{a}\sigma, \quad t' = \sqrt{a}T'$$

and

$$(2.8) \quad \zeta - \sqrt{\rho} = a\gamma_a(z) = a\frac{z-1}{\sqrt{1+a} + \sqrt{1+az}}$$

so that

$$(t - t')\zeta + \psi_a(t') = t\sqrt{\rho} + a^{\frac{3}{2}}((T - T')\gamma_a(z) + \tilde{\psi}_a(T'))$$

and therefore

$$\phi_{a,N,\hbar} = t\sqrt{\rho} + a^{\frac{3}{2}}\varphi_{a,N,\lambda}$$

with

$$(2.9) \quad \varphi_{a,N,\lambda}(T, X, T', \sigma, z) = \gamma_a(z)(T - T') + \tilde{\psi}_a(T') + \sigma(X - z) + \sigma^3/3 \\ + N(-\frac{4}{3}z^{\frac{3}{2}} + \frac{1}{\lambda}B(\lambda z^{\frac{3}{2}})),$$

where $\lambda = a^{\frac{3}{2}}/\hbar$ will be our large parameter. One may remark that $\varphi_{a,N,\lambda}$ is C^∞ in a and

$$(2.10) \quad \varphi_{0,N,\lambda} = \frac{z-1}{2}(T - T') + \frac{T'^3}{24} + \sigma(X - z) + \frac{1}{3}\sigma^3 + N(-\frac{4}{3}z^{\frac{3}{2}} + \frac{1}{\lambda}B(\lambda z^{\frac{3}{2}}))$$

and we have $z \geq \beta/2 > 0$ on the support of the symbols in our integrals.

We have now

$$(2.11) \quad \gamma_N(T, X, Y, \hbar) = \frac{\sqrt{a}}{(2\pi\hbar)^2} \int e^{i\frac{a^3}{\hbar}Y} w_N(T, X, \hbar) \eta \chi_0(\eta) d\eta,$$

and

$$(2.12) \quad w_N(T, X, \hbar) = \frac{(-i)^N \lambda}{2\pi} \int e^{i\lambda\varphi_{a,N,\lambda}} \chi_2(z) \frac{\chi_3(\sqrt{1+az})}{2\sqrt{1+az}} \chi_4(\sqrt{a}\sigma) \sigma_0(\sqrt{a}T', \hbar) dT' d\sigma dz$$

where we used $dt'dsd\zeta = \frac{a^2}{2\sqrt{1+az}} dT' d\sigma dz$ and $a^2/(2\pi\hbar) = \sqrt{a}\lambda/(2\pi)$. Finally, we set $\theta = \sqrt{a}\mu$ and $\tilde{\lambda} = \lambda/\eta = a^{\frac{3}{2}}/\hbar$. By our change of variables, the differential operator P becomes

$$P = a^{-2}Q_a \text{ with } Q_a = -\partial_X^2 - (X-1)\partial_Y^2 + 2\sqrt{1+a}\partial_T\partial_Y + a\partial_T^2,$$

and

$$Q_a(e^{i\lambda Y} f(T, X)) = e^{i\lambda Y} \lambda^2 Q_a f$$

with

$$Q_a = (\frac{1}{i\lambda}\partial_X)^2 + (X-1) - 2\sqrt{\rho}\frac{1}{i\lambda}\partial_T - a(\frac{1}{i\lambda}\partial_T)^2.$$

Our initial data at $T = 0$ is now

$$\frac{\sqrt{a}}{(2\pi\hbar)^2} \int e^{i\frac{a^3}{\hbar}(\eta Y + (X-1)\Xi)} \chi_0(\eta) \chi_1(\sqrt{a}\Xi/\eta) d\eta d\Xi,$$

and is concentrated at $Y = 0, X = 1$. The new operator Q_a has symbol

$$\sigma(Q_a) = \Xi^2 + X - 1 - 2\sqrt{\rho}\tau - a\tau^2,$$

and the positive root in τ of $\sigma(Q_a)$ at $X = 1$ is

$$\tau = \frac{\Xi^2}{\sqrt{\rho} + \sqrt{\rho + a\Xi^2}}.$$

Notice that, as τ is preserved by the flow, the bouncing angles at $X = 0$ are such that $\Xi_{\text{bounce}}^2 = 1 + 2\sqrt{\rho}\tau + a\tau^2 = 1 + \Xi_0^2 \geq 1$; we are now facing only transverse reflexions, however we aim at studying the flow for very large times.

Remark 2.10. Assuming the worst terms are $|\xi| \lesssim \sqrt{a}$, this translates into $|\Xi| \lesssim 1$ which implies τ bounded, and for a small \mathcal{Q}_a degenerates to a Schrödinger operator.

From $\partial_{T'} = \sqrt{a}\partial_{t'}$, $\partial_s = \sqrt{a}\partial_\sigma$ and $\partial_z = \frac{a}{2}(1+az)^{-\frac{1}{2}}\partial_\zeta$, we have

$$(2.13) \quad \partial_{T'}\varphi_{a,N,\lambda} = \tilde{\psi}'_a(T') + \frac{1-z}{\sqrt{\rho} + \sqrt{1+az}}$$

$$(2.14) \quad \partial_\sigma\varphi_{a,N,\lambda} = X - z + \sigma^2$$

$$(2.15) \quad \partial_z\varphi_{a,N,\lambda} = \frac{1}{2\sqrt{1+az}}(T - T' - 2\sigma\sqrt{1+az} - 4N\sqrt{z}\sqrt{1+az}(1 - \frac{3}{4}B'(\lambda z^{\frac{3}{2}}))).$$

The Lagrangian of $\tilde{\psi}_a$ is parametrized by

$$(2.16) \quad T' = -2\mu\sqrt{1+a+a\mu^2}, \quad \tilde{\psi}'_a(T') = \frac{\mu^2}{\sqrt{\rho} + \sqrt{1+a+a\mu^2}}$$

and as $1+az = \zeta^2$, and $\zeta = (1+a+\theta^2)^{1/2}$ on $\mathcal{C}_{a,N,h}$, we have

$$1 + \mu^2 = z \quad \text{on } \mathcal{C}_{a,N,h}.$$

In our new set of coordinates, the projection of $\Lambda_{a,N,h}$ onto \mathbb{R}^3 is, with $z = 1 + \mu^2$,

$$(2.17) \quad X = 1 + \mu^2 - \sigma^2$$

$$(2.18) \quad Y = 2\mu^2(\mu - \sigma)H_1(a, \mu) + \frac{2}{3}(\sigma^3 - \mu^3) + 4N(1 - \frac{3}{4}B'(\lambda z^{\frac{3}{2}}))H_2(a, \mu)$$

$$(2.19) \quad T = 2\sqrt{\rho + a\mu^2}(\sigma - \mu + 2N\sqrt{1 + \mu^2}(1 - \frac{3}{4}B'(\lambda z^{\frac{3}{2}})))$$

where $H_{1,2}$ are defined as

$$(2.20) \quad H_1(a, \mu) = \frac{\sqrt{\rho + a\mu^2}}{\sqrt{\rho} + \sqrt{\rho + a\mu^2}}$$

$$(2.21) \quad H_2(a, \mu) = \sqrt{1 + \mu^2} \frac{\frac{2}{3} + \frac{5a}{9} + \mu^2(-\frac{1}{3} + \frac{a}{9}) - \frac{4}{9}a\mu^4}{\sqrt{\rho}\sqrt{\rho + a\mu^2} + 1 + \frac{2}{3}a(1 + \mu^2)}.$$

Remark 2.11. Notice that the parameters are μ, σ and η through the λ factor in the X, Y, T parametrization of $\Lambda_{a,N,h}$.

2.5. Proof of Proposition 2.4. We already dealt with the first item. We now address the remaining two.

2.5.1. *The boundary condition.* Set

$$F_N(\zeta, \hbar) = (-i)^N e^{\frac{i}{\hbar}(-\frac{4}{3}N(\zeta^2-1)^{\frac{3}{2}} + \hbar NB((\zeta^2-1)^{\frac{3}{2}}/\hbar))} \chi_2\left(\frac{\zeta^2-1}{a}\right) \chi_3(\zeta) \hat{g}_0(\zeta/\hbar, \hbar)$$

and recall

$$v|_{x=0} = (2\pi\hbar)^{-2} \int e^{i\eta y/\hbar} \eta \chi_0(\eta) \hbar^{-2/3} e^{it\zeta/\hbar} \sum_{0 \leq N \leq C_0/\sqrt{a}} ((A_+ + A_-)(\hbar^{-2/3}(\zeta^2 - 1))) F_N d\zeta d\eta;$$

recall as well that we constructed F_N so that

$$F_N = (-1)^N \left(\frac{A_-}{A_+} \right)^N F_0,$$

which allows to cancel all middle terms in the sum to get

$$v|_{x=0} = (2\pi\hbar)^{-2} \int e^{i\eta y/\hbar} \eta \chi_0(\eta) \hbar^{-2/3} e^{it\zeta/\hbar} ((A_+(\cdots)F_0 + A_-(\cdots)F_{N_{\max}})) d\zeta d\eta.$$

Let us define

$$I_0(t, \hbar) = \int e^{i(t-t')\frac{\zeta}{\hbar}} A_+(\hbar^{-2/3}(\zeta^2 - 1)) e^{\frac{i}{\hbar}\psi_a(t')} \chi_2 \chi_3 \sigma_0 dt' d\zeta$$

$$I_{N_{\max}}(t, \hbar) = \int e^{i(t-t')\frac{\zeta}{\hbar}} A_-(\hbar^{-2/3}(\zeta^2 - 1)) e^{\frac{i}{\hbar}(-\frac{4}{3}N(\zeta^2-1)^{\frac{3}{2}} + \hbar NB((\zeta^2-1)^{\frac{3}{2}}/\hbar))} e^{\frac{i}{\hbar}\psi_a(t')} \chi_2 \chi_3 \sigma_0 dt' d\zeta.$$

Hence, it is enough to prove that

- $I_0 \in O_{C^\infty}(h^\infty)$ for $t \geq 0$, uniformly in a ;
- $I_{N_{\max}} \in O_{C^\infty}(h^\infty)$ for $t \leq 1$, uniformly in a, N_{\max} .

Start with I_0 . Using our change of scales, $I_0(t, \hbar) = J_0(a^{-\frac{1}{2}}t, \lambda)$ and

$$J_0 = \hbar^{\frac{1}{6}} a^{\frac{3}{2}} a^{-\frac{1}{4}} \int e^{i\lambda(\gamma_a(z)(T-T') + \tilde{\psi}_a(T') + \frac{2}{3}z^{\frac{3}{2}})} \chi_2(z) \chi_3(\sqrt{1+az}) z^{-\frac{1}{4}} m_0(\lambda z^{\frac{3}{2}}) \sigma_0(\sqrt{a}T', \hbar) \frac{dT' dz}{2\sqrt{1+az}}$$

where m_0 is a symbol of order 0. As we have $\partial_t = a^{-1/2}\partial_T$ and $\lambda = a^{\frac{3}{2}}/\hbar \geq \eta h^{-\frac{3}{2}\varepsilon}$ and $a \geq h^{\frac{2}{3}-\varepsilon}$, we are left to prove the following:

$$J_0(T, \lambda) \in O_{C^\infty}(\lambda^{-\infty}) \text{ for } T \geq 0, \text{ uniformly in } a.$$

We already computed the derivatives of the phase of J_0 . Recall that T' and μ are related by $T' = -2\mu\sqrt{1+a+a\mu^2}$, and $a\mu^2$ is bounded.

$$\partial_{T'}(\text{phase of } J_0) = \frac{\mu^2 + 1 - z}{\sqrt{1+az} + \sqrt{1+a+a\mu^2}}$$

$$\partial_z(\text{phase of } J_0) = \frac{T - T' + 2\sqrt{z}\sqrt{1+az}}{2\sqrt{1+az}}$$

where for the first derivative one uses $\partial_{T'} = \sqrt{a}\partial_{\nu'}$ and the identity

$$\gamma_a(z)(T - T') + \tilde{\psi}_a(T') + \frac{2}{3}z^{\frac{3}{2}} = a^{-\frac{3}{2}}((t - t')\zeta + \frac{2}{3}(\zeta^2 - 1)^{\frac{3}{2}} + \psi_a(t') - t\sqrt{1+a}).$$

The first derivative vanishes if $z = 1 + \mu^2$, and the second one vanishes if

$$T = -2\sqrt{1+a+a\mu^2}(\mu + \sqrt{1+\mu^2}) < 0.$$

As such, the phase has no critical points for $T \geq 0$. One has to be careful as the domain of integration of the (T', z) variables is very large with small a , as it is like $(-c/\sqrt{a}, c/\sqrt{a}) \times (1/2, \xi_1/a)$.

We turn to the details: for $z \leq z_0$, z is bounded. For large $|T'|$, we get $|\partial_{T'}(\text{phase})| \approx \mu^2 \approx T'^2$, therefore by integration by parts in T' we get decay. If $|T'|$ is bounded there is no critical points in (z, T') .

We are left with $z \geq z_0$, where z_0 is large. We first perform the integration in T' . There will be two critical points in T' , given by $\mu_{\pm} = \pm\sqrt{z-1}$. Denote by $J_{a,\pm}(z)$ the critical values of the phase $-\gamma_a(z)T' + \tilde{\psi}_a(T')$. We have

$$J_{a,\pm}(z) = (-\gamma_a(z)T' + \tilde{\psi}_a(T'))|_{T'=-2(\pm)\sqrt{z-1}\sqrt{1+az}}$$

and

$$\frac{d}{dz}J_{a,\pm}(z) = \pm 2\sqrt{z-1}\sqrt{1+az}\gamma'_a(z) = \pm\sqrt{z-1}.$$

We are left with the z integral,

$$\int_{z_0}^{+\infty} e^{i\lambda(\gamma_a(z)T + \frac{2}{3}z^{\frac{3}{2}} + J_{a,\pm}(z))} g_a(z, \lambda) dz,$$

where g_a is a symbol of order $-1/4$, uniformly in a : $|\partial_z^l(z^{\frac{1}{4}}g_a(z, \lambda))| \leq C_l z^{-l}$ with C_l independent of $a, z \geq z_0$, and g_a is supported in $[z_0, c/a]$ with small c (notice we used that the m_0 and χ_3 terms in J_0 are symbols of order 0, uniformly in a). We have

$$\partial_z(\gamma_a(z)T + \frac{2}{3}z^{\frac{3}{2}} + J_{a,\pm}(z)) = \sqrt{z} \pm \sqrt{z-1} + T\gamma'_a(z),$$

and for the $+$ case we may integrate by parts in z without difficulties. For the $-$ case, set

$$\mathcal{J} = \partial_z(\gamma_a(z)T + \frac{2}{3}z^{\frac{3}{2}} + J_{a,-}(z)) = \sqrt{z} - \sqrt{z-1} + \frac{T}{2\sqrt{1+az}}.$$

For $T \geq 0$, $z \in \mathbb{C}$ with $|\text{Im } z| \leq \delta|\text{Re } z|$ and $z_0 \leq \text{Re } z \leq c/a$, we have

$$|\mathcal{J}| \geq \text{Re } \mathcal{J} \geq \frac{C}{\sqrt{z}},$$

with a constant C which does not depend on a or $T \geq 0$. Hence by the Cauchy formula, $|\partial_z^l \mathcal{J}^{-1}| \leq C_l z^{\frac{1}{2}-l}$ and \mathcal{J}^{-1} is a symbol of order $1/2$, uniformly in $a, T \geq 0$. We may then conclude by integration by parts in z with the operator $\lambda^{-1}\partial_z(\mathcal{J}^{-1}\cdot)$ (if g is a symbol of order m , $\partial_z(\mathcal{J}^{-1}g)$ is a symbol of order $m - \frac{1}{2}$).

The remaining integral $I_{N_{\max}}$ may be dealt with in a similar way. In fact, the situation is easier: on $\Lambda_{a,N,\hbar} \cap \{x=0\}$ we have $s = \pm\sqrt{a+\theta^2}$ and therefore

$$t = 2\sqrt{1+a+\theta^2}(\pm\sqrt{a+\theta^2} - \theta + 2N_{\max}\sqrt{a+\theta^2}(1 - \frac{3}{4}B')) \geq 3C_0.$$

2.5.2. *The initial data.* Taking into account Lemmata 2.1 and 2.2, we are left to prove that $V_N(0, x, y, h) \in O_{C^\infty}(h^\infty)$ uniformly in $1 \leq N \leq C_0/\sqrt{a}$ for $x \geq 0$. Recall $x = \sqrt{a}X$ and from (2.17) we have

$$T = 0 \iff \sigma = \mu - 2N\sqrt{1 + \mu^2}\alpha, \quad \text{with } \alpha = 1 - \frac{3}{4}B'(\lambda z^{\frac{3}{2}}) = 1 + O\left(\frac{1}{\lambda^2 z^3}\right).$$

from which we get

$$X = 1 + \mu^2 - \sigma^2 = 1 - 4N(1 + \mu^2)\alpha(N(1 + \mu^2)\alpha - \mu)$$

and, as $N \geq 1$ and $\mu^2(\alpha^2 - 1) = \mu^2 O((\lambda z^{\frac{3}{2}})^2) = O((z - 1)/(\lambda^2 z^3)) \in O(\lambda^{-2})$,

$$4N\sqrt{1 + \mu^2}(N\sqrt{1 + \mu^2}\alpha - \mu) \geq 2(1 - O(\frac{1}{\lambda^2})).$$

For $\lambda \geq \lambda_0$, λ_0 large, we get, uniformly in N , $X \leq -1/2$ on the projection of $\Lambda_{a,N,h}$, which is what we need, $w_N(0, X, h) \in O_{C^\infty}(h^\infty)$ for $X \geq 0$, uniformly in N . We turn to the details. As before, we will proceed by integration by parts.

$$w_N(0, X, h) = \frac{\lambda}{2\pi} \int e^{i\lambda\psi} g dT' d\sigma dz,$$

where g is a symbol in σ, T', z and

$$\psi = -\gamma_a(z)T' + \tilde{\psi}_a(T') + \sigma(X - z) + \sigma^3/3 + N(-\frac{4}{3}z^{\frac{3}{2}} + \frac{B(\lambda z^{\frac{3}{2}})}{\lambda}).$$

For $z \leq z_0$, we may localize σ to a compact region as $\partial_\sigma \psi = X - z + \sigma^2 \geq \sigma^2 - z_0$ and large $|T'|$ will not be a problem. Then for T', σ, z in a compact set we get decay from the geometrical observation on the Lagrangian.

For $z \geq z_0$, we may again eliminate T' and obtain two contributions,

$$\int e^{i\lambda\psi_\pm} g_\pm d\sigma dz,$$

where

$$\psi_\pm = \pm \frac{2}{3}(z - 1)^{\frac{3}{2}} + \sigma(X - z) + \frac{\sigma^3}{3} + N(-\frac{4}{3}z^{\frac{3}{2}} + \frac{B(\lambda z^{\frac{3}{2}})}{\lambda}).$$

By integration in σ , the case $z < X - 1$ provide decay, while for $z \geq X + 1$ we have again two contributions $\pm 2/3(z - X)^{\frac{3}{2}}$. The associated phases in z are

$$\pm \frac{2}{3}(z - 1)^{\frac{3}{2}} \pm \frac{2}{3}(z - X)^{\frac{3}{2}} + N(-\frac{4}{3}z^{\frac{3}{2}} + \frac{B(\lambda z^{\frac{3}{2}})}{\lambda}),$$

for which we readily observe that they are non stationary: not only derivatives never vanish, but they increase in value with N (for $N = 1$ and the two plus signs, we deal with large values of z as we have done just above, for the boundary condition).

We are left with the contributions of $X - 1 \leq z \leq X + 1$, for which again we may reduce to compact σ as $\partial_\sigma \psi_\pm = \sigma^2 + X - z$, and we conclude by the geometric observation on the Lagrangian.

2.6. Decay for the parametrix. This section is devoted to the proof of the following result.

Theorem 2.1. *Let $\alpha < 4/7$. There exists C such that for all $h \in]0, h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0, 1]$, all $T \in]0, a^{-1/2}]$ and all $Y \in \mathbb{R}$, the following holds true*

$$(2.22) \quad \left| \sum_{0 \leq N \leq C_0/\sqrt{a}} \gamma_N(T, X, Y, h) \right| \leq C(2\pi h)^{-2} \left(\left(\frac{h}{a^{1/2}T} \right)^{1/2} + a^{1/8}h^{1/4} \right)$$

Remark 2.12. Note that, in the given range of parameters a, h , the above theorem immediately implies our main result, Theorem 1.3, after undoing the rescaling from Section 2.4.

We first observe that $\gamma_0(T, X, Y, h) = v_0(t, x, y, h)$ where v_0 is a solution of $Pv_0 = 0$ in $x > -1$ with $WF_h v_0 \subset \{\tau > 0\}$. By Proposition 2.4 (and its proof), the associated data at time $t = 0$ is a smoothed out Dirac at $x = a, y = 0$. Thus v_0 satisfies the classical dispersive estimate for the wave equation in two space dimensions and since $t = a^{1/2}T$, this implies

$$|\gamma_0(T, X, Y, h)| \leq C(2\pi h)^{-2} \left(\frac{h}{a^{1/2}T} \right)^{1/2}.$$

Thus we may assume in the proof of (2.22) that the summation is taken over $1 \leq N \leq C_0 a^{-1/2}$. Recall

$$(2.23) \quad \gamma_N(T, X, Y, h) = \frac{\sqrt{a}}{(2\pi h)^2} \int e^{i\frac{a^{3/2}}{h}\eta Y} w_N(T, X, \hbar) \eta \chi_0(\eta) d\eta$$

where the w_N are defined by

$$(2.24) \quad w_N(T, X, \hbar) = \frac{(-i)^N \lambda}{2\pi} \int e^{i\lambda \varphi_{a,N,\lambda}} \chi_2(z) \frac{\chi_3(\sqrt{1+az})}{2\sqrt{1+az}} \chi_4(\sqrt{a}\sigma) \sigma_0(\sqrt{a}T', \hbar) dT' d\sigma dz.$$

We split each w_N in two pieces, $w_N = w_{N,1} + w_{N,2}$; $w_{N,2}$ is defined by introducing an extra cutoff $\chi_5(z) \in C_0^\infty([0, z_0])$ in the integral (2.24), with $z_0 > 1$, close to 1, and $\chi_5(z) = 1$ on $[\beta/2, (1+z_0)/2]$. $w_{N,1}$ is then defined by introducing the cutoff $1 - \chi_5(z)$ in the integral (2.24). We denote by $\gamma_N = \gamma_{N,1} + \gamma_{N,2}$ the corresponding splitting using formula (2.23). The following propositions obviously imply theorem 2.1.

Proposition 2.13. *There exists C such that for all $h \in]0, h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0, 1]$, all $T \in]0, a^{-1/2}]$ and all $Y \in \mathbb{R}$, the following holds true*

$$(2.25) \quad \left| \sum_{2 \leq N \leq C_0/\sqrt{a}} \gamma_{N,1}(T, X, Y, h) \right| \leq C(2\pi h)^{-2} h^{1/3}.$$

Proposition 2.14. *There exists C such that for all $h \in]0, h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0, 1]$, all $T \in]0, a^{-1/2}]$ and all $Y \in \mathbb{R}$, the following holds true*

$$(2.26) \quad \left| \sum_{2 \leq N \leq C_0/\sqrt{a}} \gamma_{N,2}(T, X, Y, h) \right| \leq C(2\pi h)^{-2} a^{1/8} h^{1/4}.$$

Proposition 2.15. *There exists C such that for all $h \in]0, h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0, 1]$, all $T \in]0, a^{-1/2}]$ and all $Y \in \mathbb{R}$, the following holds true*

$$(2.27) \quad |\gamma_1(T, X, Y, h)| \leq C(2\pi h)^{-2} \left(\left(\frac{h}{a^{1/2}T} \right)^{1/2} + a^{1/8} h^{1/4} \right).$$

The remaining part of this section is devoted to the proof of these 3 propositions. We also include a separate section which contains useful geometric estimates, and a section where we recall useful known estimates on phase integrals.

Remark 2.16. The hypothesis $a \in [h^\alpha, a_0]$ with $\alpha < 4/7$ in theorem 2.1 will be used to see that only a few γ_N overlap with each others. We will address estimate (2.22) in the full range $a \in [h^{2/3-\epsilon}, a_0]$ in a forthcoming paper.

2.6.1. Geometric estimates. We denote in this section by $f(a, a\mu^2)$ various analytic functions defined for a and $a\mu^2$ small, with $f(a, b) \in \mathbb{R}$ for $(a, b) \in \mathbb{R}^2$. Recall that the projection of $\Lambda_{a,N,h}$ onto \mathbb{R}^3 is given by

$$(2.28) \quad \begin{aligned} X &= 1 + \mu^2 - \sigma^2 \\ Y &= 2\mu^2(\mu - \sigma)H_1 + \frac{2}{3}(\sigma^3 - \mu^3) + 4N(1 - \frac{3}{4}B'(\lambda z^{\frac{3}{2}}))H_2 \\ T &= 2\sqrt{\rho + a\mu^2}(\sigma - \mu + 2N\sqrt{1 + \mu^2}(1 - \frac{3}{4}B'(\lambda z^{\frac{3}{2}}))) \end{aligned}$$

with $z = 1 + \mu^2$ and $H_{1,2}$ of the form (see (2.20))

$$(2.29) \quad \begin{aligned} H_1 &= f_0(a, a\mu^2), \quad f_0(0, 0) = 1/2 \\ H_2(1 + \mu^2)^{-1/2} &= f_1(a, a\mu^2) + \mu^2 f_2(a, a\mu^2), \quad f_1(0, 0) = 1/3, \quad f_2(0, 0) = -1/6. \end{aligned}$$

Let us rewrite the system of equation (2.28) in the following form

$$(2.30) \quad \begin{aligned} X &= 1 + \mu^2 - \sigma^2 \\ Y &= 2\mu^2(\mu - \sigma)H_1 + \frac{2}{3}(\sigma^3 - \mu^3) + 2H_2(1 + \mu^2)^{-1/2} \left(\frac{T}{2\sqrt{\rho + a\mu^2}} - \sigma + \mu \right). \end{aligned}$$

and

$$(2.31) \quad 2N(1 - \frac{3}{4}B'(\lambda z^{\frac{3}{2}})) = (1 + \mu^2)^{-1/2} \left(\frac{T}{2\sqrt{\rho + a\mu^2}} - \sigma + \mu \right)$$

Then (2.30) and (2.31) is obviously equivalent to (2.28). For a given a and a given point $(X, Y, T) \in \mathbb{R}^3$, (2.30) is a system of two equations for the unknown (μ, σ) , and we will use the fact that (2.31) gives an equation for N . Recall that we are looking at solutions of (2.30) in the range

$$a \in [h^\alpha, a_0], \alpha < 4/7, \quad a|\mu|^2 \leq \epsilon_0, \quad 0 < T \leq a^{-1/2}, \quad X \in [0, 1]$$

with a_0, ϵ_0 small. Let us denote $R = 2(1 - 3Y/T)$.

Lemma 2.17. *Let $T \geq T_0 > 0$, $X \in [-2, 2]$, $Y \in \mathbb{R}$. There exists $\mu_j(X, Y, T, a) \in \mathbb{C}$, $j = 1, 2, 3, 4$ such that*

$$\{\mu \in \mathbb{C}, a|\mu|^2 \leq \epsilon_0, \exists \sigma \in \mathbb{C}, (\mu, \sigma) \text{ is a solution of (2.30)}\} \subset \{\mu_1, \mu_2, \mu_3, \mu_4\}$$

Moreover, there exists a function $f_*(a, a\mu^2)$ with $f_*(0, 0) = 1$, and constants $C_0, C_1, C_2 > 0$, $R_0, M_0 > 0$ such that the following holds true:

(a) If $|R| \geq R_0$, two of the $\mu_j f_*(a, a\mu_j^2)$ are in the complex disk $D(\sqrt{R}, A)$, the two others in the complex disk $D(-\sqrt{R}, A)$ with $A = C_0(1/T + \frac{a(1+|R|)}{\sqrt{|R|}})$. Moreover, one has $\sqrt{|R|} \geq 2A$.

(b) If $|R| \leq R_0$ and $|R|T \geq M_0$, two of the $\mu_j f_*(a, a\mu_j^2)$ are in the complex disk $D(\sqrt{R}, A)$, the two others in the complex disk $D(-\sqrt{R}, A)$ with $A = \frac{C_1}{T\sqrt{|R|}}$. Moreover, one has $\sqrt{|R|} \geq 2A$.

(c) If $|R| \leq R_0$ and $|R|T \leq M_0$, one has $|\mu_j| \leq C_2 T^{-1/2}$ for all j .

Proof. We first get rid of σ . The second equation in (2.30) is of the form $Y = B_0 + \sigma B_1 + \sigma^3 B_2$, thus by the first equation, we get

$$(2.32) \quad Y - B_0 = \sigma(B_1 + B_2(1 + \mu^2 - X))$$

and then the first line of (2.30) gives an equation for μ

$$(2.33) \quad (Y - B_0)^2 = (1 + \mu^2 - X)(B_1 + B_2(1 + \mu^2 - X))^2,$$

where

$$(2.34) \quad \begin{aligned} B_0 &= 2\mu^3 H_1 - 2\mu^3/3 + 2H_2(1 + \mu^2)^{-1/2} \left(\frac{T}{2\sqrt{\rho + a\mu^2}} + \mu \right) \\ B_1 &= -2\mu^2 H_1 - 2H_2(1 + \mu^2)^{-1/2} \\ B_2 &= 2/3. \end{aligned}$$

By (2.20) and (2.29) one gets through explicit computation the identity

$$(2.35) \quad \forall w, \quad f_0(0, w) + f_2(0, w) = 1/3$$

and this implies, (we use $a\mu^{2+k} = (a\mu^2)\mu^k$)

$$(2.36) \quad \begin{aligned} B_0 &= -\mu^2 T/6 f_3 + 2/3 \mu f_4 + T/3 f_5 \\ B_1 + B_2(1 + \mu^2 - X) &= -2X/3 + f_6 \end{aligned}$$

with $f_l(0, 0) = 1$ for $l = 3, 4, 5$ and $f_6(0, 0) = 0$. Let $D = -2X/3 + f_6$. We may then rewrite (2.33) as

$$(2.37) \quad P_+ P_- = 36(1 - X) \frac{D^2}{T^2}$$

with

$$(2.38) \quad P_{\pm} = (f_* \mu)^2 - \frac{2(f_* \mu)}{T} (2f_4 \mp 3D)/f_* - R', \quad R' = R + f_7, \quad f_7(0, 0) = 0$$

and $f_* = \sqrt{f_3}$, $f_*(0, 0) = 1$.

By classical arguments on perturbations of polynomial equations, (2.37) implies that for a and $a|\mu|^2$ small, the μ equation (2.33) admits at most 4 complex solutions (at most since we have the constraint $a|\mu|^2$ small). Set $\mu^* = \mu f_*(a, a\mu^2)$. Then $\mu \mapsto \mu^*$ is a holomorphic change of coordinates and $|\frac{\partial \mu^*}{\partial \mu} - 1| \leq Cte(a_0 + \epsilon_0)$. With the notation $g_{\pm} = (2f_4 \mp 3D)/f_*$, the two roots $\mu_{\pm, \epsilon}^*$ of P_{ϵ} satisfy the equation (recall that g_{ϵ} and R' are functions of $(a, a\mu^2)$)

$$(2.39) \quad \mu_{\pm, \epsilon}^* = g_{\epsilon}/T \pm \sqrt{R' + (g_{\epsilon}/T)^2}, \quad \epsilon = \pm$$

Assume first $|R| \geq R_0$ with R_0 large. Then by (2.39) and $f_7(0, 0) = 0$, there exists C_0 such that

$$(2.40) \quad \mu_{+, \epsilon}^* \in D(\sqrt{R}, A_0/2), \quad \mu_{-, \epsilon}^* \in D(-\sqrt{R}, A_0/2), \quad A_0 = C_0(1/T + \frac{a(1 + |R|)}{\sqrt{|R|}}).$$

For $\sqrt{|R|} \geq 2A_0$, the two disks $D_{\pm} = D(\pm\sqrt{R}, A_0)$ do not overlap and $\text{dist}(D_+, D_-) \geq \sqrt{|R|}$. Thus there exists C_3 such that

$$|P_+P_-(\mu)| \geq C_3|R|A_0^2 \geq C_3C_0^2|R|/T^2, \quad \forall \mu^* \in \mathbb{C} \setminus (D_+ \cup D_-),$$

and this contradicts (2.37) for $|R| \geq R_0$ large enough, and proves (a).

For $|R| \leq R_0$, and $|R|T \geq M_0$, with M_0 large, (2.40) remains true and thus we get

$$(2.41) \quad \mu_{+, \epsilon}^* \in D(\sqrt{R}, A_1/2), \quad \mu_{-, \epsilon}^* \in D(-\sqrt{R}, A_1/2), \quad A_1 = C_0(1/T + \frac{a(1 + R_0)}{\sqrt{|R|}}).$$

Set $A_2 = \frac{C}{T\sqrt{|R|}}$. Since $|R| \leq R_0$ and $T \leq a^{-1/2}$, for C large enough one has $A_2 \geq 2A_1$.

The two disks $D_{\pm} = D(\pm\sqrt{R}, A_2)$ do not overlap and $\text{dist}(D_+, D_-) \geq \sqrt{|R|}$ for $M_0 \geq 2C$. Thus one has

$$|P_+P_-(\mu)| \geq C_4A_2^2|R| = C_4C^2/T^2, \quad \forall \mu^* \in \mathbb{C} \setminus (D_+ \cup D_-),$$

and this contradicts (2.37) for C large enough, and proves (b) for M_0 large enough. Finally, for $|R| \leq R_0$, and $|R|T \leq M_0$, one has clearly by (2.39) and $T \leq a^{-1/2}$, $|\mu_{\pm, \pm}| \leq C_5T^{-1/2}$, and thus for $c \geq 2C_5$

$$\forall \mu \in \mathbb{C} \text{ such that } |\mu| \geq cT^{-1/2}, \quad |P_+P_-(\mu)| \geq C_6c^4/T^2,$$

This contradicts (2.37) for c large enough. The proof of Lemma 2.17 is complete. \square

Let us now study the equation (2.31), which provides N . Since $B'(u) \in O(u^{-2})$, $z \geq \beta/2 > 0$, $N \leq C_0a^{-1/2}$, and $a \geq h^{4/7}$, one has

$$(2.42) \quad |NB'(\lambda z^{\frac{3}{2}})| \in O(N\lambda^{-2}) = O(Nh^2/a^3) \in O(a^{-7/2}h^2) \in O(1).$$

Let $\langle \mu \rangle = (1 + \mu^2)^{1/2}$. From $\sigma^2 - \mu^2 = 1 - X$, we get for $X \in [-2, 2]$ $|\sigma - \mu|/\langle \mu \rangle \in O(1)$. Therefore (2.31) implies

$$(2.43) \quad 2N = T\Phi_a(\mu) + O(1), \quad \Phi_a(\mu) = \frac{1}{2\langle \mu \rangle \sqrt{\rho + a\mu^2}}.$$

Let $U = \{\mu \in \mathbb{C}, |\mu| \leq 0.5 \text{ or } |\operatorname{Im}(\mu)| \leq |\operatorname{Re}(\mu)|/\sqrt{3}\}$. Then $\Phi_a(\mu)$ is bounded on U and

$$(2.44) \quad |\Phi_a(\mu) - \Phi_a(\mu')| \leq \frac{C|\mu^2 - \mu'^2|}{\sup(<|\mu|>, <|\mu'|>)} \left(a + \frac{1}{<|\mu|> <|\mu'|>} \right), \quad \forall \mu, \mu' \in U.$$

Observe that for $b \in \mathbb{R}$, and $|b| \geq 2r$, the complex disk $D(b, r)$ is contained in U .

For a given point $(X, Y, T) \in [-2, 2] \times \mathbb{R} \times [0, C_0 a^{-1/2}]$, let us denote by $\mathcal{N}(X, Y, T)$ the set of integers $N \geq 1$ such that (2.28) admits at least one real solution (μ, σ, λ) with $a|\mu|^2 \leq \epsilon_0$ and $\lambda \geq \lambda_0$. We denote $\mathcal{N}^{\mathbb{C}}(X, Y, T)$ the set of complex N such that (2.28) admits at least one complex solution (μ, σ) with $\mu \in U$ and $a|\mu|^2 \leq \epsilon_0$ and $\lambda \geq \lambda_0$. Observe that $\mathcal{N}(X, Y, T)$ depends on a . For $E \subset \mathbb{N}$, $|E|$ will denote the cardinal of E . Observe that (2.43) implies for an absolute constant N_0

$$(2.45) \quad \mathcal{N}(X, Y, T) \subset [1, T/2 + N_0].$$

Lemma 2.18. *There exists a constant C_0 such that the following holds true.*

- (a) *For all $(X, Y, T) \in [0, 1] \times \mathbb{R} \times [0, a^{-1/2}]$, one has $|\mathcal{N}(X, Y, T)| \leq C_0$, and $\mathcal{N}^{\mathbb{C}}(X, Y, T)$ is contained in the union of 4 disks of radius C_0 .*
- (b) *For all $(X, Y, T) \in [0, 1] \times \mathbb{R} \times [0, a^{-1/2}]$, the subset of \mathbb{N} ,*

$$\mathcal{N}_1(X, Y, T) = \bigcup_{|Y' - Y| + |T' - T| \leq 1, |X' - X| \leq 1} \mathcal{N}(X', Y', T')$$

satisfies

$$|\mathcal{N}_1(X, Y, T)| \leq C_0.$$

Proof. We start with (a), which is a consequence of (2.43), since by lemma 2.17, for a given $(X, Y, T \geq T_0)$, there is at most 4 possible values of μ (for $T \leq T_0$ we use (2.45)).

We proceed with (b). By (2.45), we may assume $T \geq T_1$ with T_1 large. Recall $R = 2(1 - 3Y/T)$. Let (X', Y', T') such that $|Y' - Y| + |T' - T| \leq 1$, $X' \in [0, 1]$. Set $R' = 2(1 - 3Y'/T')$. One has $|R - R'| \leq C(1 + |R|)/T$. Let us first assume $|R| \geq 2R_0$, with R_0 as in lemma 2.17. Since T is large, one has $|R'| \geq R_0$, and $|R'| \simeq |R|$. Let $N' \in \mathcal{N}(X', Y', T')$ and μ' such that (2.28) holds true. By lemma 2.17 (a), one may assume $\mu'^* \in D(\sqrt{R'}, A')$. Take $\mu^* \in D(\sqrt{R}, A)$ associated to (X, Y, T) . Since μ' is real, one has $R' \geq R_0$, hence $R \geq 2R_0$, and therefore $\mu \in U$. Let $N \in \mathcal{N}^{\mathbb{C}}(X, Y, T)$ associated to μ . From $a^{1/2} \leq 1/T$ one gets

$$(2.46) \quad |\mu - \mu'| \leq C|\mu^* - \mu'^*| \leq C(A + A' + |\sqrt{R} - \sqrt{R'}|) \leq \frac{C(1 + |R|)}{T\sqrt{|R|}}.$$

By (2.43) and (2.44), this implies since $a|R| \simeq a|\mu'^2| \leq \epsilon_0$,

$$(2.47) \quad \begin{aligned} 2|N - N'| &\leq |T' - T|\Phi_a(\mu') + T|\Phi_a(\mu') - \Phi_a(\mu)| + O(1) \\ &\leq C(a + 1/|R|) \frac{(1 + |R|)}{\sqrt{|R|}} + O(1) \in O(1). \end{aligned}$$

Let us now assume $|R| \leq 2R_0$ and $T|R| \geq M_0 + 8$. From $|RT - R'T'| = |2(T - T') - 6(Y - Y')| \leq 8$, we get $|R'|T' \geq M_0$. We may thus apply lemma 2.17 (b). Let $N' \in \mathcal{N}(X', Y', T')$ and $\mu' \in \mathbb{R}$ such that (2.28) holds true. Since μ' is real one has $R' > 0$, thus $R'T' > M_0$, and

this implies $R > 0$ (take M_0 large). Moreover one has $|R - R'| \leq C(1 + |R|)/T$, $|R'| \leq 3R_0$, and also $|R'| \simeq |R|$. By the same argument as above, we get now $|\mu - \mu'| \leq \frac{C(1+R_0)}{T\sqrt{R}}$, and since $|\mu| + |\mu'| \leq C\sqrt{|R|}$, one gets $|\mu^2 - \mu'^2| \leq C(1 + R_0)/T$ and thus by (2.43) and (2.44)

$$(2.48) \quad 2|N - N'| \leq CT(a + O(1))(1 + R_0)/T \in O(1).$$

Finally, for $|R| \leq 2R_0$ and $T|R| \leq M_0 + 8$, one has $T'|R'| \leq M_0 + 16$, thus by part (c) of lemma 2.17, one has $|\mu'_j| \leq CT^{1/2}$, $|\mu_j| \leq CT^{-1/2}$, and thus we get in that case $|\mu^2 - \mu'^2| \leq C/T$ thus (2.48) holds also true in that case. Since $N \in \mathcal{N}^{\mathbb{C}}(X, Y, T)$, (2.47), (2.48), and part (a) of our lemma imply (b). The proof of our lemma is complete. \square

2.6.2. Phase integrals. We first recall the following lemma, for which we refer to [16].

Lemma 2.19. *Let $K \subset \mathbb{R}$ a compact set, and $a(\xi, \lambda)$ a classical symbol of degree 0 in $\lambda \geq 1$ with $a(\xi, \lambda) = 0$ for $\xi \notin K$. Let $k \geq 2$, $c_0 > 0$ and $\Phi(\xi)$ a phase function such that*

$$\sum_{2 \leq j \leq k} |\Phi^{(j)}(\xi)| \geq c_0, \quad \forall \xi \in K.$$

Then, there exists C such that

$$\left| \int e^{i\lambda\Phi(\xi)} a(\xi, \lambda) d\xi \right| \leq C\lambda^{-1/k}, \quad \forall \lambda \geq 1$$

Moreover, the constant C depends only on c_0 and on a upper bound of a finite number of derivatives of $\Phi^{(2)}$, a in a neighborhood of K .

The next lemma will be of importance to us. As such, we have included its proof for the sake of the reader while not claiming any novelty.

Let $H(\xi)$ be a smooth function defined in a neighborhood of $(0, 0)$ in \mathbb{R}^2 , such that $H(0) = 0$ and $\nabla H(0) = 0$. We assume that the Hessian H'' satisfies $\text{rank}(H''(0)) = 1$ and $\nabla \det(H'')(0) \neq 0$. Then the equation $\det(H'')(\xi) = 0$ defines a smooth curve \mathcal{C} near $0 \in \mathbb{R}^2$ with $0 \in \mathcal{C}$. Let $s \rightarrow \xi(s)$ be a smooth parametrization of \mathcal{C} , with $\xi(0) = 0$, and define the curve $X(s)$ in \mathbb{R}^2 by

$$X(s) = H'(\xi(s)).$$

Lemma 2.20. *Let $K = \{\xi \in \mathbb{R}^2, |\xi| \leq r\}$, and $a(\xi, \lambda)$ a classical symbol of degree 0 in $\lambda \geq 1$ with $a(\xi, \lambda) = 0$ for $\xi \notin K$. Set for $x \in \mathbb{R}^2$ close to 0*

$$(2.49) \quad I(x, \lambda) = \int e^{i\lambda(x \cdot \xi - H(\xi))} a(\xi, \lambda) d\xi.$$

Then for $r > 0$ small enough, the following holds true:

(a) *If $X'(0) \neq 0$, there exists C such that for all x close to 0*

$$|I(x, \lambda)| \leq C\lambda^{-5/6}.$$

(b) *If $X'(0) = 0$ and $X''(0) \neq 0$ there exists C such that for all x close to 0*

$$|I(x, \lambda)| \leq C\lambda^{-3/4}.$$

Moreover, if a is elliptic at $\xi = 0$, there exists C' such that

$$|I(0, \lambda)| \geq C' \lambda^{-3/4}.$$

Proof. By a linear change of coordinates in ξ , we may assume $H(\xi) = \xi_1^2/2 + O(\xi^3)$. Set $\Phi(x, \xi) = x\xi - H(\xi)$. Then $\Phi'_{\xi_1}(x, \xi) = x - H'_{\xi_1}(\xi)$. Therefore, there exists a unique non degenerate critical point $\xi_1^c(x, \xi_2)$ in the variable ξ_1 , and the critical value $\Psi(x, \xi_2)$ satisfies

$$G(x, \xi_2) = \Psi'_{\xi_2}(x, \xi_2) = x_2 - H'_{\xi_2}(\xi_1^c(x, \xi_2), \xi_2),$$

and by stationary phase in ξ_1 , one has

$$I(x, \lambda) = \lambda^{-1/2} \int e^{i\lambda\Psi(x, \xi_2)} b(x, \xi_2, \lambda) d\xi_2.$$

By lemma 2.19, it remains to prove:

- (a') If $X'(0) \neq 0$, there exists $c_0 > 0$ such that for all (x, ξ_2) close to $(0, 0)$ $|\partial_{\xi_2}^2 G| \geq c_0$.
- (b') If $X'(0) = 0$ and $X''(0) \neq 0$, there exists $c_0 > 0$ such that for all (x, ξ_2) close to $(0, 0)$ $|\partial_{\xi_2}^3 G| \geq c_0$, and in the case a elliptic the lower bound at $x = 0$.

Let us prove (a'). Since $G(x, \xi_2)$ is smooth and r small, it is sufficient to prove $\partial_{\xi_2}^2 G(0, 0) \neq 0$. The Taylor expansion of H at order 3 reads as follows

$$H(\xi) = \xi_1^2/2 + a\xi_1^3 + b\xi_1^2\xi_2 + c\xi_1\xi_2^2 + d\xi_2^3 + O(\xi^4).$$

Thus one has $\xi_1^c(0, \xi_2) = -c\xi_2^2 + O(\xi_2^3)$ and we get $-G(0, \xi_2) = 3d\xi_2^2 + O(\xi_2^3)$. Thus $\partial_{\xi_2}^2 G(0, 0) \neq 0$ is equivalent to $d \neq 0$.

On the other hand, one has $\det H''(\xi) = 2c\xi_1 + 6d\xi_2 + O(\xi^2)$, and since by hypothesis $\nabla \det(H'')(0) \neq 0$, one has $(c, d) \neq (0, 0)$. Moreover, one has

$$X(s) = H'(\xi(s)) = (\xi_1(s) + O(s^2), O(s^2))$$

and therefore $X'(0) \neq 0$ is equivalent to $\xi_1'(0) \neq 0$. This in turn is equivalent to the fact that ξ_1 is a parameter on \mathcal{C} , which is equivalent to $d \neq 0$.

Let us now prove (b'). Since $X'(0) = 0$, we get $d = 0$ and therefore $c \neq 0$. Now, ξ_2 is a parameter on \mathcal{C} , and we have $\xi_1 \in O(\xi_2^2)$ on \mathcal{C} . We will use a Taylor expansion of H at order 4, but since $\xi_1^c(0, \xi_2)$ is quadratic in ξ_2 , and $\xi_1(s)$ quadratic in s , we will just need the ξ_2^4 term, i.e

$$H(\xi) = \xi_1^2/2 + a\xi_1^3 + b\xi_1^2\xi_2 + c\xi_1\xi_2^2 + e\xi_2^4 + \dots + O(\xi^5).$$

Then we get $\det H''(\xi) = 2c\xi_1 + 4(3e - c^2)\xi_2^2 + O(\xi^3)$. Therefore $\xi_1 = 2(c - 3e/c)\xi_2^2 + O(\xi_2^3)$ is an equation for \mathcal{C} , and we get that $X''(0) \neq 0$ is equivalent to $X_1''(0) \neq 0$. this in turn is equivalent to $c^2 \neq 2e$. On the other hand, we easily get $-G(0, \xi_2) = (4e - 2c^2)\xi_2^3 + O(\xi_2^4)$. Finally, for $\alpha \neq 0$ and $b(\xi_2, \lambda)$ a symbol of degree 0 elliptic at $\xi_2 = 0$, and supported in $|\xi_2| \leq r$ with r small enough, one has clearly

$$\left| \int e^{i\lambda(\alpha\xi_2^4 + O(\xi_2^5))} b(\xi_2, \lambda) d\xi_2 \right| \geq C' \lambda^{-1/4}$$

which completes the proof. \square

2.6.3. *Proof of proposition 2.13.* Recall

(2.50)

$$w_{N,1}(T, X, \hbar) = \frac{(-i)^N \lambda}{2\pi} \int e^{i\lambda\varphi_{a,N,\lambda}} \chi_2(z) \frac{\chi_3(\sqrt{1+az})}{2\sqrt{1+az}} \chi_4(\sqrt{a}\sigma) \sigma_0(\sqrt{a}T', \hbar) (1-\chi_5)(z) dT' d\sigma dz$$

where the phase $\varphi_{a,N,\lambda}$ is defined by (see (2.9))

$$(2.51) \quad \varphi_{a,N,\lambda}(T, X, T', \sigma, z) = \gamma_a(z)(T - T') + \tilde{\psi}_a(T') + \sigma(X - z) + \sigma^3/3 + N(-\frac{4}{3}z^{\frac{3}{2}} + \frac{1}{\lambda}B(\lambda z^{\frac{3}{2}})).$$

For $\epsilon_j = \pm$, define

$$(2.52) \quad \Phi_{N,\epsilon_1,\epsilon_2}(T, X, z; a, \lambda) = \gamma_a(z)T + 2\epsilon_1/3(z-1)^{3/2} + 2\epsilon_2/3(z-X)^{3/2} - N(\frac{4}{3}z^{\frac{3}{2}} - \frac{1}{\lambda}B(\lambda z^{\frac{3}{2}})).$$

Lemma 2.21. *The following identity holds true*

$$(2.53) \quad w_{N,1}(T, X, \hbar) = \sum_{\epsilon_1, \epsilon_2} \int e^{i\lambda\Phi_{N,\epsilon_1,\epsilon_2}} \Theta_{\epsilon_1,\epsilon_2}(z; a, \lambda) dz + R_{N,a}(T, X, \hbar)$$

where $\Theta_{\epsilon_j}(z; a, \lambda)$ are smooth functions of z with support in $[(1+z_0)/2, (\zeta_1^2-1)/a]$ (remark that $\zeta_1 > 1$ is an upper bound for the support of χ_3). Moreover,

$$|z^l \partial_z^l \Theta_{\epsilon_j}| \leq C_l z^{-1/2} \text{ with } C_l \text{ independent of } a, \lambda.$$

The remainder $R_{N,a}(T, X, \hbar)$ is $O_{C^\infty}(\hbar^\infty)$ for $X \in [0, 1], T \in [0, a^{-1/2}]$, uniformly in a, N .

Proof. The proof is a simple application of stationary phase in (T', σ) in the integral (2.50). Recall $z \geq (1+z_0)/2 > 1 \geq X$ on the support of $(1-\chi_5)(z)$, and $z \leq (\zeta_1^2-1)/a$ on the support of $\chi_3(\sqrt{1+az})$. The σ integral is equal to

$$(2.54) \quad J_1 = \int e^{i\lambda(\sigma^3/3 - \sigma(z-X))} \chi_4(\sqrt{a}\sigma) d\sigma = (z-X)^{1/2} \int e^{i\lambda(z-X)^{3/2}(s^3/3-s)} \chi_4(\sqrt{a}(z-X)^{1/2}s) ds.$$

One has $\sqrt{a}(z-X)^{1/2} \leq \sqrt{\zeta_1^2-1}$. Thus, by stationary phase near the two critical points $s = \pm 1$ and integration by part in s elsewhere, we get

$$(2.55) \quad J_1 = \lambda^{-1/2}(z-X)^{-1/4}(e^{2/3i\lambda(z-X)^{3/2}}b_+ + e^{-2/3i\lambda(z-X)^{3/2}}b_-) + O(\lambda^{-\infty}(z-X)^{-\infty})$$

where $b_\pm(\sqrt{a}(z-X)^{1/2}, \lambda(z-X)^{3/2})$ are symbols of degree 0 in the (large) parameter $\lambda(z-X)^{3/2}$. Next, the T' integral is equal to

$$(2.56) \quad J_2 = \int e^{i\lambda(-\gamma_a(z)T' + \tilde{\psi}_a(T'))} \sigma_0(\sqrt{a}T', \hbar) dT'.$$

Recall that

$$T' = -2\mu\sqrt{1+a+a\mu^2}, \quad \partial_{T'}(-\gamma_a(z)T' + \tilde{\psi}_a(T')) = \frac{\mu^2 + 1 - z}{\sqrt{1+az} + \sqrt{1+a+a\mu^2}}.$$

Thus we get two distinct critical points $T'_\pm = \mp 2\sqrt{z-1}\sqrt{1+az}$. The associated critical values are $-\gamma_a(z)T'_\pm + \tilde{\psi}_a(T'_\pm) = \pm 2/3(z-1)^{3/2}$. As before for the σ integral, we perform the change of variable $T' = s\sqrt{z-1}$, in order to have the two critical points $s_\pm = \mp\sqrt{1+az}$ uniformly at finite distance in z . One has $\sqrt{a}(z-1)^{1/2} \leq \sqrt{\zeta_1^2-1}$, and using (2.7) and (2.8) we get, again by stationary phase near the two critical points s_\pm and integration by part in s elsewhere,

$$(2.57) \quad J_2 = \lambda^{-1/2}(z-1)^{-1/4}(e^{2/3i\lambda(z-1)^{3/2}}c_+ + e^{-2/3i\lambda(z-1)^{3/2}}c_-) + O(\lambda^{-\infty}(z-1)^{-\infty})$$

where $c_\pm(\sqrt{a}(z-1)^{1/2}, \lambda(z-1)^{3/2})$ are symbols of degree 0 in the large parameter $\lambda(z-1)^{3/2}$. By (2.55) and (2.57), one gets that formula (2.53) holds true with symbols

$$(2.58) \quad \Theta_{\epsilon_1, \epsilon_2}(z; a, \lambda) = \frac{(-i)^N \chi_2(z)(1 - \chi_5)(z) \chi_3(\sqrt{1+az}) c_{\epsilon_1} b_{\epsilon_2}}{4\pi \sqrt{1+az} (z-1)^{1/4} (z-X)^{1/4}}$$

which completes the proof of Lemma 2.21. \square

Let

$$(2.59) \quad \begin{aligned} \gamma_{N,1,\epsilon_1,\epsilon_2}(T, X, Y, h) &= \frac{\sqrt{a}}{(2\pi h)^2} \int e^{i\frac{a^{3/2}}{h}\eta Y} w_{N,1,\epsilon_1,\epsilon_2}(T, X, \hbar) \eta \chi_0(\eta) d\eta \\ w_{N,1,\epsilon_1,\epsilon_2}(T, X, \hbar) &= \int e^{i\lambda\Phi_{N,\epsilon_1,\epsilon_2}} \Theta_{\epsilon_1,\epsilon_2}(z; a, \lambda) dz. \end{aligned}$$

In order to prove proposition 2.13, we are reduced to proving the following inequality:

$$(2.60) \quad \left| \sum_{2 \leq N \leq C_0/\sqrt{a}} \gamma_{N,1,\epsilon_1,\epsilon_2}(T, X, Y, h) \right| \leq C(2\pi h)^{-2} h^{1/3}$$

with a constant C independent of $h \in]0, h_0]$, $a \in [h^\alpha, a_0]$, $X \in [0, 1]$, $T \in [0, a^{-1/2}]$.

For convenience, we take $Z = z^{3/2}$ as a new variable of integration so that

$$(2.61) \quad w_{N,1,\epsilon_1,\epsilon_2}(T, X, \hbar) = \int e^{i\lambda\Phi_{N,\epsilon_1,\epsilon_2}} \tilde{\Theta}_{\epsilon_1,\epsilon_2}(Z; a, \lambda) dZ;$$

$\tilde{\Theta}_{\epsilon_1,\epsilon_2}(Z; a, \lambda)$ are now smooth functions of Z with support in $[((1+z_0)/2)^{3/2}, ((\zeta_1^2-1)/a)^{3/2}]$. Since $dz = 2Z^{-1/3}dZ/3$, we get $|Z^l \partial_Z^l \tilde{\Theta}| \leq C_l Z^{-2/3}$ with C_l independent of a, λ . One has

$$\begin{aligned}
(2.62) \quad \partial_Z \Phi_{N,\epsilon_1,\epsilon_2} &= \frac{2}{3} \left(H_{a,\epsilon_1,\epsilon_2}(T, X; Z) - 2N \left(1 - \frac{3}{4} B'(\lambda Z) \right) \right) \\
H_{a,\epsilon_1,\epsilon_2} &= Z^{-1/3} \left(\frac{T}{2} (1 + aZ^{2/3})^{-1/2} + \epsilon_1 (Z^{2/3} - 1)^{1/2} + \epsilon_2 (Z^{2/3} - X)^{1/2} \right) \\
\partial_Z H_{a,\epsilon_1,\epsilon_2} &= \frac{1}{3} Z^{-4/3} \left(-\frac{T}{2} (1 + aZ^{2/3})^{-3/2} (1 + 2aZ^{2/3}) \right. \\
&\quad \left. + \epsilon_1 (Z^{2/3} - 1)^{-1/2} + \epsilon_2 X (Z^{2/3} - X)^{-1/2} \right).
\end{aligned}$$

We will first prove that (2.60) holds true in the case $(\epsilon_1, \epsilon_2) = (+, +)$. From (2.62), we get that the equation $\partial_Z H_{a,+,+}(Z) = 0$ admits a unique solution $Z_q = Z_q^+(T, X, a) > 1$, such that

$$\begin{aligned}
(2.63) \quad &\lim_{T \rightarrow \infty} Z_q^+(T, X, a) = 1 \quad \text{uniformly in } X, a \\
0 &> \frac{9}{2} Z_q^{5/3} \partial_Z^2 H_{a,+,+}(Z_q) = -\frac{aT}{2} (1 + aZ_q^{2/3})^{-5/2} (1/2 - aZ_q^{2/3}) \\
&\quad - 1/2 (Z_q^{2/3} - 1)^{-3/2} - X/2 (Z_q^{2/3} - X)^{-3/2}.
\end{aligned}$$

Therefore, the function $H_{a,+,+}(Z)$ is strictly increasing on $[1, Z_q[$, and strictly decreasing on $]Z_q, \infty[$. Observe that

$$(2.64) \quad H_{a,+,+}(1) = \frac{T}{2} (1 + a)^{-1/2} + (1 - X)^{1/2}, \quad \lim_{Z \rightarrow \infty} H_{a,+,+}(Z) = 2.$$

For all k one has

$$(2.65) \quad \sup_{Z \geq 1} |\partial_Z^k (NB'(\lambda Z))| \leq C_k N \lambda^{-2} Z^{-(k+2)} \leq C'_k \hbar^2 a^{-7/2} \leq C''_k h^\nu, \quad \nu = 2 - 7\alpha/2 > 0.$$

Let $T_0 \gg 1$. We first prove that (2.60) holds true for $T \in [0, T_0]$. Since $H_{a,+,+}(Z) \leq C(1 + T)$, for $N \geq N(T_0) = C(1 + T_0)$, one gets $|\partial_Z \Phi_{N,+,+}(Z)| \geq c_0 N$ with $c_0 > 0$, and $|\partial_Z^k \partial_Z \Phi_{N,+,+}(Z)| \leq c_k N Z^{-k}$ for $k \geq 1$. Therefore, by integration by parts in Z in (2.61) with the operator $L(\Theta) = \lambda^{-1} \partial_Z ((\partial_Z \Phi_{N,+,+})^{-1} \Theta)$, one gets an extra factor $(\lambda N Z)^{-1}$ at each iteration. Thus, we get $w_{N,1,+,+} \in O(N^{-\infty} \lambda^{-\infty})$, and this implies

$$(2.66) \quad \sup_{T \leq T_0, X \in [0,1], Y \in \mathbb{R}} \left| \sum_{N(T_0) \leq N \leq C_0/\sqrt{a}} \gamma_{N,1,\epsilon_1,\epsilon_2}(T, X, Y, h) \right| \in O(h^\infty).$$

Next, for $T \in [0, T_0]$ and $2 \leq N \leq N(T_0)$, one may estimate the sum in (2.60) by the sup of each term. But in that case, we know by (2.62), (2.63), and (2.65) that there exists at most a critical point of order 2 near $Z = Z_q$ for $\Phi_{N,+,+}$, and

$$|\partial_Z \Phi_{N,+,+}| + |\partial_Z^2 \Phi_{N,+,+}| + |\partial_Z^3 \Phi_{N,+,+}| \geq c > 0.$$

Moreover, by the second item of (2.64), and $N \geq 2$, one has a positive lower bound for $|\partial_Z \Phi_{N,+,+}(Z)|$ for large values of Z ; thus, large values of Z yield $O(\lambda^{-\infty})$ contributions

to $w_{N,1,+,+}$, and eventually the worst contribution to $w_{N,1,+,+}$ will be the critical point of order 2 near $Z = Z_q$. This provides

$$|w_{N,1,+,+}(T, X, \hbar)| \leq C\lambda^{-1/3} \text{ with } C \text{ independent of } T \in [0, T_0], X \in [0, 1].$$

Since $a^{1/2}\lambda^{-1/3} = \hbar^{1/3} = (h/\eta)^{1/3}$, we get

$$(2.67) \quad \sup_{T \leq T_0, X \in [0,1], Y \in \mathbb{R}} \left| \sum_{2 \leq N \leq N(T_0)} \gamma_{N,1,\epsilon_1,\epsilon_2}(T, X, Y, h) \right| \leq C(T_0)(2\pi h)^{-2} h^{1/3}.$$

Next we prove that that (2.60) holds true for $T \in [T_0, a^{-1/2}]$. As before, we may assume $N \leq C_1 T$, with C_1 large, the contribution of the sum $C_1 T \leq N \leq C_0/\sqrt{a}$ being negligible. Recall that we have $Z \geq Z_0 = ((1+z_0)/2)^{3/2} > 1$ on the support of $\tilde{\Theta}_{+,+}$ in formula (2.61). By the first item of (2.63), one may choose T_0 large enough so that $Z_q^+(T, X, a) \leq (1+Z_0)/2 < Z_0$ for all $T \geq T_0$. By the last item of (2.62), increasing T_0 if necessary, and using (2.65), we may assume with a constant $c > 0$

$$(2.68) \quad |\partial_Z^2 \Phi_{N,+,+}(Z)| \geq cTZ^{-4/3}, \quad \forall Z \geq Z_0, \forall T \geq T_0, \forall N \leq C_0 a^{-1/2}$$

Therefore, on the support of $\tilde{\Theta}_{+,+}$, the phase $\Phi_{N,+,+}$ admits at most one critical point $Z_c = Z_c(T, X, N, \lambda, a)$ and this critical point is non degenerate. Since $N \geq 2$, from the first two items of (2.62) we get $Z_c^{1/3} \leq T$, and this implies $Z_c^{1/3} \simeq T/N$. If T/N is bounded, Z_c is bounded, and since $\partial_Z \Phi \leq -c < 0$ for large Z , we get by stationary phase

$$|w_{N,1,+,+}(T, X, \hbar)| \leq C\lambda^{-1/2} T^{-1/2} \text{ with } C \text{ independent of } N.$$

If T/N is large, then we perform the change of variable $Z = s(T/N)^3$ in (2.61); the unique critical point s_c remains in a fixed compact interval of $]0, \infty[$, one has $\partial_s \Phi \leq -c(T/N)^3 < 0$ for s large and also

$$\partial_s^k \tilde{\Theta}_{+,+}(s(T/N)^3, a, \lambda) \leq C_k (N/T)^2 s^{-2/3-k}.$$

Thus, by stationary phase

$$(2.69) \quad \forall T \in [T_0, a^{-1/2}], \quad \sup_{2 \leq N \leq C_1 T} \sup_{X \in [0,1]} |w_{N,1,+,+}(T, X, \hbar)| \leq C\lambda^{-1/2} T^{-1/2}.$$

By lemma 2.18, we know that for any given $M = (X, Y, T)$ there is at most C_0 values of N such that the projection of $\mathbf{\Lambda}_{a,N,h}$ intersect the ball of radius 1 centered at M ; therefore, we will prove that the previous arguments imply

$$(2.70) \quad \sup_{T \in [T_0, a^{-1/2}], X \in [0,1], Y \in \mathbb{R}} \left| \sum_{2 \leq N \leq C_0/\sqrt{a}} \gamma_{N,1,+,+}(T, X, Y, h) \right| \leq C(T_0)(2\pi h)^{-2} (a^{-1/4} h^{1/2})$$

and since $a^{-1/4} h^{1/2} \leq h^{1/3}$, we will get that (2.60) holds true. Let us now explain more precisely how one can estimate the sum in (2.70) by the supremum over N .

Let $G_N(T, X, \lambda, a) = \Phi_{N,+,+}(T, X, Z_c(T, X, N, \lambda, a); a, \lambda)$. The stationary phase at the critical point $Z_c = Z_c(T, X, N, \lambda, a)$ in (2.61) gives

$$(2.71) \quad w_{N,1,+,+}(T, X, \hbar) = \lambda^{-1/2} T^{-1/2} e^{i\lambda G_N(T, X, \lambda, a)} \psi_N(T, X; a, \lambda).$$

By (2.55), (2.57), (2.58), we know that the $\psi_N(T, X; a, \lambda)$ are symbols of degree 0 in λ , and $\partial_\lambda^k \psi_N \leq C_k \lambda^{-k}$ with C_k independent of $2 \leq N \leq C_1 T$. This gives with $\tilde{\lambda} = a^{3/2}/h = \lambda/\eta$

$$(2.72) \quad \gamma_{N,1,+,+}(T, X, Y, h) = \frac{T^{-1/2} h^{1/2} a^{-1/4}}{(2\pi h)^2} \times \int e^{i\tilde{\lambda}\eta(Y+G_N(T,X,\tilde{\lambda}\eta,a))} \psi_N(T, X; a, \tilde{\lambda}\eta) \eta^{1/2} \chi_0(\eta) d\eta.$$

This is an integral with large parameter $\tilde{\lambda}$ and phase

$$L_N(T, X, Y, \eta, \tilde{\lambda}) = \eta(Y + G_N(T, X, \tilde{\lambda}\eta)).$$

By construction, the equation

$$\partial_\eta L_N = Y + G_N(T, X, \lambda, a) + \lambda \partial_\lambda G_N(T, X, \lambda, a) = 0$$

implies that (X, Y, T) belongs to the projection of $\mathbf{\Lambda}_{a,N,h}$ on \mathbb{R}^3 . Let $T \in [T_0, a^{-1/2}]$, $X \in [0, 1]$ and $Y \in \mathbb{R}$ be given. For $N \notin \mathcal{N}_1(X, Y, T)$, one has therefore $\partial_\eta L_N(T', X', Y', \eta, \tilde{\lambda}) \neq 0$ for all λ and all $X' \in [0, 1]$, $|Y' - Y| + |T' - T| \leq 1$. This implies, since $\partial_\eta L_N$ is linear in Y , $|\partial_\eta L_N(T, X, Y, \eta, \tilde{\lambda})| \geq 1$. Moreover, one has, with C_k independent of $N, T, X, \eta, \lambda, a$

$$(2.73) \quad |\partial_\eta^k (\partial_\eta L_N)| \leq C_k.$$

To prove (2.73), we just use that $\partial_\lambda Z_c$ satisfies

$$\partial_\lambda Z_c \partial_Z^2 \Phi_{N,+,+}(Z_c) = -\partial_\lambda \partial_Z \Phi_{N,+,+}(Z_c) = N Z_c B''(\lambda Z_c)$$

and thus from (2.62) and (2.65), we get for all $k \geq 1$, $(\eta \partial_\eta)^k Z_c = (\lambda \partial_\lambda)^k Z_c \in O(h^\nu)$. Then (2.73) follows from

$$\lambda \partial_\lambda G_N(T, X, \lambda) = \lambda (\partial_\lambda \Phi_{N,+,+})(T, X, Z_c; a, \lambda) = \frac{N}{\lambda} (-B(\lambda Z_c) + \lambda Z_c B'(\lambda Z_c)).$$

Therefore, by integration by parts in η in (2.72), we get

$$(2.74) \quad \sup_{T \in [T_0, a^{-1/2}], X \in [0, 1], Y \in \mathbb{R}} \left| \sum_{N \notin \mathcal{N}_1(X, Y, T)} \gamma_{N,1,+,+}(T, X, Y, h) \right| \in O(h^\infty).$$

Finally, by lemma 2.18, one has $|\mathcal{N}_1(X, Y, T)| \leq C_0$, and therefore, we get from (2.74) and (2.69) that (2.70) holds true.

Next, we show that (2.60) holds true for $(\epsilon_1, \epsilon_2) = (-, +)$. In that case, from the last item of (2.62) and $X \in [0, 1]$, one gets $\partial_Z H_{a,-,+} < 0$ for $T > 0$. Therefore the function $H_{a,-,+}(Z)$ decreases on $[1, \infty[$, from $H_{a,-,+}(1) = \frac{T(1+a)^{-1/2}}{2} + (1-X)^{1/2}$ to $H_{a,-,+}(\infty) = 0$. The equation $\partial_Z \Phi_{N,-,+} = 0$ admits an unique solution Z_c , and this critical point is non degenerate. We can thus argue as we have done before for the $(+, +)$ case.

Finally, one sees that the case $(\epsilon_1, \epsilon_2) = (+, -)$ is similar to the $(+, +)$ case, and the case $(\epsilon_1, \epsilon_2) = (-, -)$ is similar to the $(-, +)$ case. We leave the details to the reader.

The proof of Proposition 2.13 is complete.

2.6.4. *Proof of proposition 2.14.* Recall that

$$(2.75) \quad w_{N,2}(T, X, \hbar) = \frac{(-i)^N \lambda}{2\pi} \int e^{i\lambda\varphi_{a,N,\lambda}} \chi_2(z) \frac{\chi_3((1+az)^{\frac{1}{2}})}{2(1+az)^{\frac{1}{2}}} \\ \times \chi_4(a^{\frac{1}{2}}\sigma) \sigma_0(a^{\frac{1}{2}}T', \hbar) \chi_5(z) dT' d\sigma dz$$

as well as $\partial_{T'}\varphi_{a,N,\lambda} = \frac{\mu^2+1-z}{\sqrt{1+az}+\sqrt{1+a+a\mu^2}}$ and $\partial_\sigma\varphi_{a,N,\lambda} = X - z + \sigma^2$.

Let $K = \{T' = 0, \sigma \in [-1, 1], z = 1\}$, let ω be a small neighborhood of K and $\chi_6(T', \sigma, z) \in C_0^\infty(\omega)$ equal to 1 near K . Since for z in the support of the integral (2.75), one has $z \in [\beta/2, z_0]$ with $z_0 > 1$ close to 1, decreasing $z_0 > 1$ if necessary, we get by integration by parts in T', σ

$$(2.76) \quad w_{N,2}(T, X, \hbar) = \frac{(-i)^N \lambda}{2\pi} \int e^{i\lambda\varphi_{a,N,\lambda}} \chi(T', \sigma, z; a, \hbar) dT' d\sigma dz + O(\lambda^{-\infty}) \\ \chi(T', \sigma, z; a, \hbar) = \chi_2(z) \frac{\chi_3(\sqrt{1+az})}{2\sqrt{1+az}} \chi_4(\sqrt{a}\sigma) \sigma_0(\sqrt{a}T', \hbar) \chi_5(z) \chi_6(T', \sigma, z)$$

with $O(\lambda^{-\infty})$ uniform in T, X, N, a . Moreover, $\chi(T', \sigma, z; a, \hbar)$ is a classical symbol of degree 0 in \hbar , with support $(T', \sigma, z) \in \omega$ and a is just a harmless parameter in χ .

We first perform the integration with respect to z in (2.76). Recall

$$(2.77) \quad \varphi_{a,N,\lambda}(T, X, T', \sigma, z) = \gamma_a(z)(T - T') + \tilde{\psi}_a(T') + \sigma(X - z) + \sigma^3/3 \\ + N(-\frac{4}{3}z^{\frac{3}{2}} + \frac{1}{\lambda}B(\lambda z^{\frac{3}{2}})) \\ \partial_z\varphi_{a,N,\lambda}(T, X, T', \sigma, z) = \frac{T - T'}{2(1+az)^{1/2}} - \sigma - 2Nz^{1/2}(1 - \frac{3}{4}B'(\lambda z^{3/2})).$$

Thus, $\varphi_{a,N,\lambda}$ admits a unique critical point $z_c(T, T', \sigma, a, \lambda)$, and we are just interested with values of the parameters such that z_c is close to 1. With $u = (\frac{T-T'}{2} - \sigma)/2N$, this means u close to 1. Since σ is close to $[-1, 1]$ and $N \geq 1$, we may thus assume $\tilde{T} = T/4N$ close to $[1/2, 3/2]$, say $\tilde{T} \in [1/4, 2]$. We denote by $g(\tilde{T}, T', \sigma; a, N, \lambda)$ various functions which are classical symbols of degree 0 in λ and with parameters a, N ; in particular, with $w = (\tilde{T}, T', \sigma)$, for all α , there exists C_α independent of a, N, λ such that $|\partial_w^\alpha g| \leq C_\alpha$ for all $\tilde{T} \in [1/4, 2]$ and all (T', σ) close to $(0, 0)$.

We denote by $f_k(a, \tilde{T}, T'/N, \sigma/N)$ functions which are homogeneous of degree k in $(T'/N, \sigma/N)$. The notation \mathcal{O}_j means any function of the form $f = \sum_{k \geq j} f_k$. We will

use the following functions

$$\begin{aligned}
 F_0 &= \frac{2\tilde{T}^2}{1 + \sqrt{1 + 4a\tilde{T}^2}} \\
 G_0^{-1} &= F_0^{1/2}(1 + aF_0)^{1/2} \left(\frac{1}{F_0} + \frac{a}{1 + aF_0} \right) \\
 H_0 &= \frac{1 - F_0}{\sqrt{1 + a} + \sqrt{1 + aF_0}} \\
 F_1 &= -\frac{G_0}{N}(T'/2 + \sigma(1 + aF_0)^{1/2}).
 \end{aligned}
 \tag{2.78}$$

Lemma 2.22. (a) *One has*

$$z_c = F_0 + F_1 + \mathcal{O}_2 + g_0/\lambda^2.$$

(b) *The critical value $\Psi_{a,N,\lambda}(\tilde{T}, X, T', \sigma) = \varphi_{a,N,\lambda}(T, X, T', \sigma, z_c)$ is equal to*

$$\begin{aligned}
 \Psi_{a,N,\lambda} &= (X - F_0)\sigma + \sigma^3/3 + H_0T' + \tilde{\psi}_a(T') \\
 &+ \frac{G_0}{2N(1 + aF_0)^{1/2}}(\sigma(1 + aF_0)^{1/2} + T'/2)^2 \\
 &- \frac{1}{12N^2}(T'/2 + \sigma)^3 + aN\mathcal{O}_3 + g_1/\lambda^2 \\
 &+ N(4\tilde{T}\gamma_a(F_0) - \frac{4}{3}F_0^{3/2} + g_2(\tilde{T}, a, N, \lambda)/\lambda^2).
 \end{aligned}
 \tag{2.79}$$

Proof. (a) The equation for z_c when $\lambda = \infty$ is

$$z^{1/2}(1 + az)^{1/2} = \tilde{T} - \frac{1}{2N}(T'/2 + \sigma(1 + az)^{1/2}).$$

The solution of this equation is clearly of the form $z = \sum f_k(a, \tilde{T}, T'/N, \sigma/N)$ with $f_0 = F_0$ solution of $F_0(1 + aF_0) = \tilde{T}^2$, and we get F_1 by a Taylor expansion at order 1. Then (a) is a consequence of the implicit function theorem applied to

$$z^{1/2}(1 + az)^{1/2}(1 - \frac{3}{4}B'(\lambda z^{3/2})) = \tilde{T} - \frac{1}{2N}(T'/2 + \sigma(1 + az)^{1/2}).$$

To prove part (b), one may of course insert the formula for z_c into the definition of $\varphi_{a,N,\lambda}$. Another way is to use

$$\begin{aligned}
 \partial_\sigma(\Psi_{a,N,\lambda} - \sigma X - \sigma^3/3 - \tilde{\psi}_a) &= -z_c \\
 \partial_{T'}(\Psi_{a,N,\lambda} - \sigma X - \sigma^3/3 - \tilde{\psi}_a) &= -\gamma_a(z_c).
 \end{aligned}
 \tag{2.80}$$

Using part (a), this system is easily seen to be integrable and yields formula (2.79) up to an integration constant which is easy to compute when $T' = \sigma = 0$. Moreover, when $a = 0$ and $\lambda = \infty$, one has $z_c^{1/2} = \tilde{T} - \frac{1}{2N}(T'/2 + \sigma)$; therefore, one can easily compute the value of the critical value when $a = 0, \lambda = \infty$, and this provides the first two terms on the second line of (2.79). The proof of our lemma is complete. \square

From $aT \in O(a^{1/2})$ and $N\lambda^{-2} \in O(h^\nu)$, we get $\partial_z^2 \varphi_{a,N,\lambda} = -Nz^{-1/2} + O(a^{1/2} + h^\nu)$ for any z close to 1. Decreasing a_0 and h_0 if necessary, we get by stationary phase,

$$(2.81) \quad \int e^{i\lambda\varphi_{a,N,\lambda}} \chi(T', \sigma, z; a, \hbar) dT' d\sigma dz = \frac{2\pi}{\sqrt{\lambda N}} \int e^{i\lambda\Psi_{a,N,\lambda}} \tilde{\chi}(\tilde{T}, T', \sigma; 1/N, a, \hbar) dT' d\sigma.$$

Here, $\tilde{\chi}$ is a classical symbol of degree 0 in \hbar , with harmless parameters $a, 1/N$. Let us define $\tilde{\gamma}_{N,2}(T, X, Y, h)$ by the following formula, where $\tilde{\lambda} = a^{3/2}/h = \lambda/\eta$:

$$(2.82) \quad \tilde{\gamma}_{N,2}(T, X, Y, h) = \int e^{i\tilde{\lambda}\eta(Y + \Psi_{a,N,\eta\tilde{\lambda}})} \tilde{\chi}\eta^{3/2} \chi_0(\eta) dT' d\sigma d\eta.$$

By (2.76) and (2.81), Proposition 2.14 is clearly a consequence of the following estimate:

$$(2.83) \quad \sum_{4N/T \in [1/4, 2]} \frac{1}{\sqrt{N}} |\tilde{\gamma}_{N,2}(T, X, Y, h)| \leq C \tilde{\lambda}^{-3/4}$$

with C independent of $X \in [0, 1]$, $T \in]0, a^{-1/2}]$, $Y \in \mathbb{R}$, $a \in [h^\alpha, a_0]$ and $\tilde{\lambda} \in [\tilde{\lambda}_0, \infty[$ with a_0 small and $\tilde{\lambda}_0$ large.

Lemma 2.23. *For all k there exist C_k independent of $X \in [0, 1]$, $T \in]0, a^{-1/2}]$, $Y \in \mathbb{R}$, $a \in [h^\alpha, a_0]$ and $\tilde{\lambda} \in [\tilde{\lambda}_0, \infty[$ such that*

$$(2.84) \quad \sum_{\substack{N \notin \mathcal{N}_1(X, Y, T) \\ 4N/T \in [1/4, 2]}} \frac{1}{\sqrt{N}} |\tilde{\gamma}_{N,2}(T, X, Y, h)| \leq C_k \tilde{\lambda}^{-k}.$$

Proof. Recall that $T' = -2\mu\sqrt{1+a+a\mu^2}$. Let us define the functions (see (2.28))

$$(2.85) \quad \begin{aligned} \mathcal{X} &= 1 + \mu^2 - \sigma^2 \\ \mathcal{Y} &= 2\mu^2(\mu - \sigma)H_1 + \frac{2}{3}(\sigma^3 - \mu^3) + 4N(1 - \frac{3}{4}B'(\eta\tilde{\lambda}(1 + \mu^2)^{\frac{3}{2}}))H_2 \\ \mathcal{T} &= 2\sqrt{\rho + a\mu^2}(\sigma - \mu + 2N\sqrt{1 + \mu^2}(1 - \frac{3}{4}B'(\eta\tilde{\lambda}(1 + \mu^2)^{\frac{3}{2}}))). \end{aligned}$$

There exists a universal constant C_0 , with $\Phi = \eta(Y + \phi_{a,N,\eta\tilde{\lambda}})$, such that

$$(2.86) \quad |X - \mathcal{X}| + |Y - \mathcal{Y}| + |T - \mathcal{T}| \leq C_0(|\Phi'_\eta| + |\Phi'_{T'}| + |\Phi'_\sigma| + |\Phi'_z|).$$

The left hand side of (2.86) does not depend of z . Since $G = \eta(Y + \phi_{a,N,\eta\tilde{\lambda}})$ is a critical value of Φ with respect to z , we get

$$(2.87) \quad |X - \mathcal{X}| + |Y - \mathcal{Y}| + |T - \mathcal{T}| \leq C_0(|G'_\eta| + |G'_{T'}| + |G'_\sigma|).$$

Consider any given (X, Y, T) . Then for $N \notin \mathcal{N}_1(X, Y, T)$, we have $|x - \mathcal{X}| + |y - \mathcal{Y}| + |t - \mathcal{T}| \neq 0$ for all values of $\mu, \sigma, \eta, \tilde{\lambda}$, all $|x - X| \leq 1$, all $|y - Y| \leq 1$, all $|t - T| \leq 1$. This implies $|X - \mathcal{X}| + |Y - \mathcal{Y}| + |T - \mathcal{T}| \geq 1$. Therefore, for all values of $\mu, \sigma, \eta, \tilde{\lambda}$, and $N \notin \mathcal{N}_1(X, Y, T)$ we get

$$|G'_\eta| + |G'_{T'}| + |G'_\sigma| \geq 1/C_0.$$

From (2.79), using $\eta\partial_\eta = \lambda\partial_\lambda$, $N/\lambda^2 \in O(h^\nu)$ and the fact that any function of type $N\mathcal{O}_3$ is in $O(N^{-2})$, we get that all the derivatives of G with respect to η, T', σ are uniformly bounded. By integration by part in (2.82) we thus get $|\tilde{\gamma}_{N,2}| \leq C_k \tilde{\lambda}^{-k}$ for all k , with C_k independent of $X \in [0, 1]$, $T \in]0, a^{-1/2}]$, $Y \in \mathbb{R}$, and $N \notin \mathcal{N}_1(X, Y, T)$. The proof of Lemma 2.23 is complete. \square

From Lemma 2.23, and since $|\mathcal{N}_1(X, Y, T)|$ is uniformly bounded by lemma 2.18, we get that (2.83) will be a consequence of

$$(2.88) \quad \forall N \text{ with } 4N/T \in [1/4, 2], \quad \frac{1}{\sqrt{N}} \left| \int e^{i\lambda\Psi_{a,N,\lambda}} \tilde{\chi} dT' d\sigma \right| \leq C\lambda^{-3/4}.$$

Here and after, we denote by C any constant which is independent of $N \geq 1$, $X \in [0, 1]$, $T \in]0, a^{-1/2}]$, $a \in [h^\alpha, a_0]$ and $\lambda \in [\lambda_0, \infty[$ with a_0 small and λ_0 large.

Observe that we can now replace the phase $\Psi_{a,N,\lambda}$ by the phase

$$(2.89) \quad \psi_{a,N,\lambda} = (X - F_0)\sigma + \sigma^3/3 + H_0 T' + \tilde{\psi}_a(T') \\ + \frac{G_0}{2N(1 + aF_0)^{1/2}} (\sigma(1 + aF_0)^{1/2} + T'/2)^2 - \frac{1}{12N^2} (T'/2 + \sigma)^3 + aN\mathcal{O}_3$$

since by (2.79) the difference $\Psi_{a,N,\lambda} - (\psi_{a,N,\lambda} + g_1/\lambda^2)$ does not depend on T', σ , and $e^{ig_1/\lambda}$ is a classical symbol of order 0 in λ . We set $\underline{\chi} = e^{ig_1/\lambda} \tilde{\chi}$ and recall $\hbar = a^{3/2}/\lambda$. Then $\underline{\chi}(T', \sigma; \tilde{T}, a, N; \lambda)$ is a classical symbol of degree 0 in $\lambda \geq \lambda_0$, compactly supported in (T', σ) close to $\{0\} \times [-1, 1]$. Moreover, for all α , there exists C_α independent of a, N and $\tilde{T} \in [1/4, 2]$, such that $\sup_{(T', \sigma, \lambda)} |\partial_{(T', \sigma)}^\alpha \underline{\chi}| \leq C_\alpha$.

Lemma 2.24. *There exists C such that for all $N \geq \lambda^{1/3}$,*

$$(2.90) \quad \frac{1}{\sqrt{N}} \left| \int e^{i\lambda\psi_{a,N,\lambda}} \underline{\chi} dT' d\sigma \right| \leq C\lambda^{-5/6}.$$

Proof. It is sufficient to prove that, for all $N \geq \lambda^{1/3}$,

$$(2.91) \quad \left| \int e^{i\lambda\psi_{a,N,\lambda}} \underline{\chi} dT' d\sigma \right| \leq C\lambda^{-2/3}.$$

Set $X - F_0 = -A\lambda^{-2/3}$, $H_0 = -B\lambda^{-2/3}$ and perform a change of variable in (2.91): $T' = \lambda^{-1/3}x$, $\sigma = \lambda^{-1/3}y$. We are reduced to proving

$$(2.92) \quad \left| \int e^{iG} \underline{\chi}(\lambda^{-1/3}x, \lambda^{-1/3}y, \dots) dx dy \right| \leq C$$

with a phase G of the following form

$$(2.93) \quad G = -Ay + y^3/3 - Bx + \lambda\tilde{\psi}_a(\lambda^{-1/3}x) + \frac{G_0\lambda^{1/3}}{2N(1 + aF_0)^{1/2}} (y(1 + aF_0)^{1/2} + x/2)^2 \\ + N^{-2}(x, y)^3 f(a, \tilde{T}, \lambda^{-1/3}x/N, \lambda^{-1/3}y/N).$$

Then (2.92) is an oscillatory integral over a domain of integration of size $\lambda^{1/3}$. Parameters $F_0, G_0, \lambda^{1/3}/N$ are bounded, and the main point is to prove that the constant C is uniform in $(A, B) = (r \cos \theta, r \sin \theta)$ with $r \leq c_0 \lambda^{2/3}$. Recall $\lambda \tilde{\psi}_a(\lambda^{-1/3}x) = \frac{x^3}{24(1+a)^{3/2}}(1 + O(ax^2\lambda^{-2/3}))$. One has

$$(2.94) \quad \begin{aligned} \partial_x G &= -B + \lambda^{2/3} \tilde{\psi}'_a(\lambda^{-1/3}x) + O((x, y)) + O(1) \\ \partial_y G &= -A + y^2 + O((x, y)) + O(1). \end{aligned}$$

Moreover, since $\underline{\chi}$ is compactly supported in (T', σ) one has, with C_α independent of \tilde{T}, a, N, λ ,

$$\sup_{(x, y)} |\partial_{(x, y)}^\alpha \underline{\chi}(\lambda^{-1/3}x, \lambda^{-1/3}y, \dots)| \leq C_\alpha (1 + |x| + |y|)^{-|\alpha|}.$$

Therefore, for any r_0 , the oscillatory integral (2.92) is clearly bounded for $0 \leq r \leq r_0$ (integrate by part for large (x, y)).

For $r \in [r_0, c_0 \lambda^{2/3}]$, we rescale variables $(x, y) = r^{1/2}(x', y')$, and we set $G = r^{3/2}G'$ and $\chi'(x', y', \dots) = \underline{\chi}(r^{1/2}\lambda^{-1/3}x', r^{1/2}\lambda^{-1/3}y', \dots)$. Observe that since $r^{1/2}\lambda^{-1/3}$ is bounded, we still have decay estimates

$$\sup_{(x', y')} |\partial_{(x', y')}^\alpha \chi'| \leq C_\alpha (1 + |x'| + |y'|)^{-|\alpha|}.$$

We have to prove

$$(2.95) \quad r \left| \int e^{ir^{3/2}G'} \chi' dx' dy' \right| \leq C.$$

First, the critical points of G' satisfy

$$(2.96) \quad \begin{aligned} \partial_{x'} G' &= -\cos \theta + r^{-1} \lambda^{2/3} \tilde{\psi}'_a(r^{1/2} \lambda^{-1/3} x') + r^{-1/2} O((x', y')) + r^{-1} O(1) \\ \partial_{y'} G' &= -\sin \theta + y'^2 + r^{-1/2} O((x', y')) + r^{-1} O(1). \end{aligned}$$

Let $F(u)$ denote a smooth function near $u = 0$, we then have

$$r^{-1} \lambda^{2/3} \tilde{\psi}'_a(r^{1/2} \lambda^{-1/3} x') = \frac{x'^2}{8(1+a)^{3/2}} (1 + ar \lambda^{-2/3} x'^2 F(a^{1/2} r^{1/2} \lambda^{-1/3} x')).$$

By (2.96), the contribution of large (x', y') to (2.95) is $O(r^{-\infty})$, and we may localize the integral on a compact set in (x', y') . For $|\sin \theta| \geq 0.1$, and r_0 large, we have two distinct non degenerate critical points $y'_\pm = \pm |\sin \theta|^{1/2} + O(r^{-1/2})$ in y' with critical values $G'_\pm(x', \dots)$ and thus we get by stationary phase

$$(2.97) \quad r \int e^{ir^{3/2}G'} \chi' dx' dy' = r^{1/4} \left(\int e^{ir^{3/2}G'_+} \chi'_+ dx' + \int e^{ir^{3/2}G'_-} \chi'_- dx' \right).$$

Moreover, one has

$$\partial_{x'} G'_\pm(x', \dots) = \partial_{x'} G'(x', y'_\pm, \dots) = -\cos \theta + r^{-1} \lambda^{2/3} \tilde{\psi}'_a(r^{1/2} \lambda^{-1/3} x') + O(r^{-1/2})$$

and this imply $|\partial_{x'}^3 G'_\pm| \geq c_0 > 0$. Thus, we get

$$\left| \int e^{ir^{3/2}G'_\pm} \chi'_\pm dx' \right| \leq C(r^{3/2})^{-1/3} = Cr^{-1/2}.$$

Therefore, (2.95) holds true, and in fact, we have the following better estimate in the range $r \geq 1$:

$$(2.98) \quad r \left| \int e^{ir^{3/2}G'} \chi' dx' dy' \right| \leq Cr^{-1/4}.$$

If $\sin \theta$ is close to 0, then we first perform the stationary phase in x' , and we use the same arguments. This completes the proof of Lemma 2.24. \square

Therefore, we can now assume that $\frac{\lambda}{N^3} = \Lambda \geq 1$, and take Λ as our new large parameter. We set $X - F_0 = -pN^{-2}$, $H_0 = -qN^{-2}/2$ and we change variables in (2.88): $T' = -2x/N$, $\sigma = -y/N$. We will prove

$$(2.99) \quad \left| \int e^{i\Lambda \mathcal{G}_a} \underline{\chi}(x/N, y/N, \dots) dx dy \right| \leq C\Lambda^{-3/4}$$

with a phase \mathcal{G}_a which takes the following form

$$(2.100) \quad \mathcal{G}_a = py - y^3/3 + qx + N^3 \tilde{\psi}_a(-2x/N) + \frac{G_0}{2(1+aF_0)^{1/2}} (y(1+aF_0)^{1/2} + x)^2 \\ + \frac{1}{12N^2} (x+y)^3 + aN^{-2} (x, y)^3 f(a, \tilde{T}, \frac{x}{N^2}, \frac{y}{N^2}).$$

Observe that, for large N , (2.99) implies a better estimate than (2.83); more precisely, (2.99) is equivalent to

$$(2.101) \quad \frac{1}{\sqrt{N}} \left| \int e^{i\lambda \Psi_{a,N,\lambda}} \tilde{\chi} dT' d\sigma \right| \leq CN^{-1/4} \lambda^{-3/4}.$$

The above estimate is of course compatible with (2.90) for $N \simeq \lambda^{-1/3}$.

Recall, see (2.7), that $\tilde{\psi}_a(T') = \frac{T'^3}{24(1+a)^{3/2}} (1 + O(aT'^2))$. From (2.93) we get

$$(2.102) \quad \partial_x \mathcal{G}_a = q - x^2 + \frac{G_0}{(1+aF_0)^{1/2}} (y(1+aF_0)^{1/2} + x) + \frac{1}{4N^2} (x+y)^2 + aO((x, y)^2) \\ \partial_y \mathcal{G}_a = p - y^2 + G_0 (y(1+aF_0)^{1/2} + x) + \frac{1}{4N^2} (x+y)^2 + aO((x, y)^2)$$

and

$$(2.103) \quad \partial_{x,x}^2 \mathcal{G}_a = -2x + \frac{G_0}{(1+aF_0)^{1/2}} + \frac{1}{2N^2} (x+y) + aO((x, y)) \\ \partial_{x,y}^2 \mathcal{G}_a = G_0 + \frac{1}{2N^2} (x+y) + aO((x, y)) \\ \partial_{y,y}^2 \mathcal{G}_a = -2y + G_0 (1+aF_0)^{1/2} + \frac{1}{2N^2} (x+y) + aO((x, y)).$$

Thus we get the value of the hessian $\mathcal{H}_N(x, y; \tilde{T}, a)$

$$(2.104) \quad \begin{aligned} \mathcal{H}_N(x, y; \tilde{T}, a) &= \det \begin{pmatrix} \partial_{x,x}^2 \mathcal{G}_a & \partial_{x,y}^2 \mathcal{G}_a \\ \partial_{y,x}^2 \mathcal{G}_a & \partial_{y,y}^2 \mathcal{G}_a \end{pmatrix} \\ &= -2G_0(x+y) + 4xy - \frac{1}{N^2}(x+y)^2 + aO((x, y)). \end{aligned}$$

Lemma 2.25. *There exists r_0 and C such that for all (p, q) with $|(p, q)| \geq r_0$*

$$(2.105) \quad \left| \int e^{i\Lambda \mathcal{G}_a} \underline{\chi}(x/N, y/N, \dots) dx dy \right| \leq C\Lambda^{-5/6}.$$

Proof. Set $(q, p) = (r \cos \theta, r \sin \theta)$ with $r \geq r_0$ large. Let $\chi \in C_0^\infty(|(x, y)| < c)$ with small c and $\chi = 1$ near 0. Then by (2.102) we get for all k by integration by part in (x, y)

$$\left| \int e^{i\Lambda \mathcal{G}_a} \chi(r^{-1/2}(x, y)) \underline{\chi}(x/N, y/N, \dots) dx dy \right| \leq C_k r^{-k} \Lambda^{-k}.$$

For the remaining term, we perform the change of variable $(x, y) = r^{1/2}(x', y')$ and we set $\mathcal{G}'_a = r^{-3/2} \mathcal{G}_a$. It remains to prove

$$(2.106) \quad \left| r \int e^{ir^{3/2}\Lambda \mathcal{G}'_a} (1 - \chi)(x', y') \underline{\chi}(r^{1/2}x'/N, r^{1/2}y'/N, \dots) dx' dy' \right| \leq C\Lambda^{-5/6}.$$

Observe that since $(1 - \chi)(x', y') = 0$ near 0, $(1 - \chi)(x', y') = 1$ for $|(x', y')| \geq c$, and since $\underline{\chi}(u, v, \dots)$ is compactly supported in (u, v) , we still have

$$\sup_{(x', y')} |\partial_{(x', y')}^\alpha (1 - \chi)(x', y') \underline{\chi}(r^{1/2}x'/N, r^{1/2}y'/N, \dots)| \leq C_\alpha (1 + |x'| + |y'|)^{-|\alpha|}.$$

The phase \mathcal{G}'_a is of the form

$$\mathcal{G}'_a = \sin \theta y' - \frac{y'^3}{3} + \cos \theta x' - \frac{x'^3}{3} + \frac{1}{12N^2}(x' + y')^3 + r^{-1/2}O((x', y')^2) + aO((x', y')^3)$$

where $O((x', y')^k)$ means any function of the form $x'^j y'^l f(r^{1/2}x'/N, r^{1/2}y'/N, a, N)$ with f smooth uniformly in a, N and $j + l = k$. Thus for small a and large r_0 , we may localize the integral (2.106) to a compact set in (x', y') (integrate by part). The hessian of \mathcal{G}'_a is equal to $4x'y' - \frac{1}{N^2}(x' + y')^2 + O(r^{-1/2} + a)$. Thus for $N \geq 2$, a small and r_0 large, the set on which the hessian vanishes defines a smooth curve Γ outside $(x', y') = (0, 0)$, which is close to the union of the two lines $c(x' + y') \pm (x' - y') = 0$, $c^2 = \frac{N^2 - 1}{N^2} \in [3/4, 1]$. Moreover, one has

$$(2.107) \quad \begin{aligned} \partial_{x'} \mathcal{G}'_a &= \cos \theta - x'^2 + \frac{1}{4N^2}(x' + y')^2 + O(r^{-1/2} + a) \\ \partial_{y'} \mathcal{G}'_a &= \sin \theta - y'^2 + \frac{1}{4N^2}(x' + y')^2 + O(r^{-1/2} + a). \end{aligned}$$

The contribution of points (x', y') outside Γ to the left hand side of (2.106) is estimated by $O(r^{-1/2}\Lambda^{-1})$ by the usual phase stationary theorem. To estimate the contribution of

points (x', y') close to Γ , we use lemma 2.20. For any value of θ , one gets easily that the hypothesis of part (a) of lemma 2.20 holds true, and this gives the estimate

$$|\int e^{ir^{3/2}\Lambda\mathcal{G}'_a}(1-\chi)(x', y')\underline{\chi}(r^{1/2}x'/N, r^{1/2}y'/N, \dots)dx'dy'| \leq C(r^{3/2}\Lambda)^{-5/6} = r^{-5/4}\Lambda^{-5/6}$$

which gives the bound $Cr^{-1/4}\Lambda^{-5/6}$ on the right hand side of (2.106). The proof of Lemma 2.25 is complete. \square

We can now assume $|(p, q)| \leq r_0$. There exists $c > 0$ independent of $N \geq 2$ such that

$$(2.108) \quad \forall (x, y) \in \mathbb{R}^2, \quad |x^2 - \frac{1}{4N^2}(x+y)^2| + |y^2 - \frac{1}{4N^2}(x+y)^2| \geq c(x^2 + y^2).$$

Thus by integration by part, (2.102) yields that large values of (x, y) gives a contribution $O(\Lambda^{-\infty})$ to the integral (2.99). We can then replace $\underline{\chi}(x/N, y/N, \dots)$ by a symbol $\chi(x, y, \dots)$ compactly supported in the ball $B = |(x, y)| \leq R$ with R large. We are left to prove

$$(2.109) \quad |\int e^{i\Lambda\mathcal{G}_a}\chi dx dy| \leq C\Lambda^{-3/4}.$$

Uniformly in $N \geq 1$ and \tilde{T} near $[1/2, 3/2]$ and (x, y) near B , we have

$$(2.110) \quad \mathcal{G}_a = \mathcal{G}_0 + O(a) \\ \mathcal{G}_0 = qx - x^3/3 + py - y^3/3 + \frac{\tilde{T}}{2}(x+y)^2 + \frac{1}{12N^2}(x+y)^3.$$

Note that the hessian of \mathcal{G}_a is $\mathcal{H}_a = -2\tilde{T}(x+y) + 4xy - \frac{1}{N^2}(x+y)^2 + O(a)$. Therefore the set $\mathcal{Z}_a = \{(x, y); \mathcal{H}_a(x, y) = 0\}$ is, for small a , a smooth curve in B which is close to the parabola $-2\tilde{T}(x+y) - (x-y)^2 = 0$ for $N = 1$, and close to the hyperbole $-2\tilde{T}(x+y) + 4xy - \frac{1}{N^2}(x+y)^2 = 0$ for $N \geq 2$. Moreover,

$$(2.111) \quad \partial_x \mathcal{G}_a = q - \mathcal{X}(x, y, a); \quad \mathcal{X}(x, y, a) = x^2 - \tilde{T}(x+y) - \frac{1}{4N^2}(x+y)^2 + O(a) \\ \partial_y \mathcal{G}_a = p - \mathcal{Y}(x, y, a); \quad \mathcal{Y}(x, y, a) = y^2 - \tilde{T}(x+y) - \frac{1}{4N^2}(x+y)^2 + O(a)$$

It remains to use Lemma 2.20 near any point (q, p) , $|(p, q)| \leq r_0$. If (q, p) is not in the image of \mathcal{Z}_a by the map $(\mathcal{X}, \mathcal{Y})$, then near (q, p) , the estimate (2.109) holds true with a factor $C\Lambda^{-1}$ on the right hand side by the usual stationary phase theorem. If (q, p) is in the image of \mathcal{Z}_a by the map $(\mathcal{X}, \mathcal{Y})$, but $(q, p) \neq (0, 0)$, then one easily verifies that part (a) of lemma 2.20 applies, and this gives near (q, p) the estimate (2.109) with a factor $C\Lambda^{-5/6}$ on the right hand side. Finally, near $(q, p) = (0, 0)$, one has (x, y) near $(0, 0)$, and one easily verifies that part (b) of lemma 2.20 applies, and therefore (2.109) holds true.

This concludes the proof of Proposition 2.14.

Remark 2.26. Since the symbol χ of degree 0 is elliptic at $(x, y) = (0, 0)$, the estimate (2.109) is optimal. To see this point, it is sufficient to apply part (b) of lemma 2.20 at $(p, q) = (0, 0)$. Observe that by (2.78), $(p, q) = (0, 0)$ is equivalent to $X = F_0 = 1$ and $T = 4N\sqrt{1+a}$, i.e equivalent to $x = a, t = 4N\sqrt{a(1+a)}$ which are precisely the times

where a swallowtail occurs in the wave front set of the Green function. This, and (2.101), proves Theorem 1.4.

2.6.5. Proof of Proposition 2.15. In order to prove Proposition 2.15, we use as before the splitting $\gamma_1 = \gamma_{1,1} + \gamma_{1,2}$. First, we prove

$$(2.112) \quad |\gamma_{1,1}(T, X, Y, h)| \leq C(2\pi h)^{-2} \left(\left(\frac{h}{a^{1/2}T} \right)^{1/2} + h^{1/3} \right).$$

Going through the proof of Proposition 2.13, we notice only one difference between the case $N = 1$ and $N \geq 2$: namely for $T \in]0, T_0]$, when we estimate $\int e^{i\lambda\Phi_{1,+,+}} \Theta_{+,+}(z; a\lambda) dz$, we may have a critical point associated to a very large value of z . Let $\chi(z) \in C_0^\infty([z_1, \infty[)$ with z_1 large, and set

$$(2.113) \quad J = \int e^{i\lambda\Phi_{1,+,+}} \Theta_{+,+}(z; a\lambda) \chi(z) dz.$$

We will prove

$$a^{1/2}|J| \leq Ca^{1/2}\lambda^{-1/2}T^{-1/2}$$

which clearly gives the first term on the right of the inequality (2.112). One has

$$(2.114) \quad \begin{aligned} \partial_z \Phi_{1,+,+} &= \frac{T}{2}(1 + az)^{-1/2} - \frac{z^{-1/2}}{2}(1 + X) + O(z^{-3/2}) \\ \partial_z^2 \Phi_{1,+,+} &= \frac{-Ta}{4}(1 + az)^{-3/2} + \frac{z^{-3/2}}{4}(1 + X) + O(z^{-5/2}). \end{aligned}$$

Therefore, to get a large critical point z_c , T must be small (recall $T \leq T_0$ means $t \leq a^{1/2}T_0$). One has then $z_c^{-1/2} \simeq T$ and from (2.114) we get $\partial_z^2 \Phi_{1,+,+}(z_c) \simeq T^3$. Recall that $\Theta_{+,+}(z, a, \lambda)$ is a classical symbol in z of degree $-1/2$, thus $T^{-1}s^{1/2}\Theta_{+,+}(T^{-2}s, a, \lambda)$ is a symbol of degree 0 in $s \geq s_0 > 0$, uniformly in $T \in]0, T_0]$. Therefore, if we perform the change of variable $z = T^{-2}s$ in (2.113), we get $|J| \leq C\lambda^{-1/2}T^{-1/2}$ by stationary phase.

It remains to prove that the inequality (2.26) holds true for $\gamma_{1,2}$. The only place where $N \geq 2$ gets used in the proof of Proposition 2.14 is Lemma 2.25 and inequality (2.108). But for $N = 1$, since $\underline{\chi}(x, y, \dots)$ is compactly supported in (x, y) , we do not need the inequality (2.108). Moreover, we get from (2.102) that the phase \mathcal{G}_a has no critical points on the support of $\underline{\chi}$ for $|(p, q)| \geq r_0$ if r_0 is large, and this implies for $|(p, q)| \geq r_0$

$$(2.115) \quad \left| \int e^{i\Lambda\mathcal{G}_a} \underline{\chi}(x, y, \dots) dx dy \right| \in O(\Lambda^{-\infty}).$$

In fact, Lemma 2.25 is telling us that the constant C in the right of (2.105) is uniform for large N . The proof of Proposition 2.15 is now complete.

3. PARAMETRIX FOR $0 < a \leq h^{1/2}$

We will write the initial data with the help of gallery modes, which we first describe in connection with the spectral analysis of our Laplace operator. We describe the corresponding solutions of the wave operator. We then estimate their $L^\infty(\Omega)$ norm for tangent initial directions by using Sobolev embedding, taking advantage of the size of the Fourier support.

We deal with the non-tangent initial directions by constructing a crude parametrix, relying partly on gallery modes and the asymptotics of the Airy function.

3.1. Whispering gallery modes. Let $\Omega = \{(x, y) \in \mathbb{R}^2 | x > 0, y \in \mathbb{R}\}$ denote the half-space \mathbb{R}_+^2 with the Laplacian given by $\Delta_D = \partial_x^2 + (1+x)\partial_y^2$ with Dirichlet boundary condition on $\partial\Omega$. Taking the Fourier transform in the y -variable gives

$$(3.1) \quad -\Delta_{D,\eta} = -\partial_x^2 + (1+x)\eta^2.$$

For $\eta \neq 0$, $-\Delta_{D,\eta}$ is a self-adjoint, positive operator on $L^2(\mathbb{R}_+)$ with compact resolvent. Indeed, the potential $V(x, \eta) = (1+x)\eta^2$ is bounded from below, it is continuous and $\lim_{x \rightarrow \infty} V(x, \eta) = \infty$. Thus one can consider the form associated to $-\partial_x^2 + V(x, \eta)$,

$$Q(u) = \int_{x>0} |\partial_x v|^2 + V(x, \eta) |v|^2 dx, \quad D(Q) = H_0^1(\mathbb{R}_+) \cap \{v \in L^2(\mathbb{R}_+), (1+x)^{1/2}v \in L^2(\mathbb{R}_+)\},$$

which is clearly symmetric, closed and bounded from below by a positive constant c . If $c \gg 1$ is chosen such that $-\Delta_{D,\eta} + c$ is invertible, then $(-\Delta_{D,\eta} + c)^{-1}$ sends $L^2(\mathbb{R}_+)$ in $D(Q)$ and we deduce that $(-\Delta_{D,\eta} + c)^{-1}$ is also a (self-adjoint) compact operator. The last assertion follows from the compact inclusion

$$D(Q) = \{v | \partial_x v, (1+x)^{1/2}v \in L^2(\mathbb{R}_+), v(0) = 0\} \hookrightarrow L^2(\mathbb{R}_+).$$

We deduce that there exists a base of eigenfunctions v_k of $-\Delta_{D,\eta}$ associated to a sequence of eigenvalues $\lambda_k(\eta) \rightarrow \infty$. From $-\Delta_{D,\eta} v = \lambda v$ we obtain $\partial_x^2 v = (\eta^2 - \lambda + x\eta^2)v$, $v(0, \eta) = 0$ and after a suitable change of variables we find that a basis of $L^2([0, \infty[)$ orthonormal eigenfunctions is given by eigenfunctions

$$(3.2) \quad e_k(x, \eta) = f_k \frac{\eta^{1/3}}{k^{1/6}} Ai(\eta^{2/3}x - \omega_k),$$

where $(-\omega_k)_k$ denote the zeros of Airy's function in decreasing order and where f_k are constants so that $\|e_k(\cdot, \eta)\|_{L^2([0, \infty[)} = 1$ for every $k \geq 1$, and the f_k remain in a fixed compact subset of $]0, \infty[$. The corresponding eigenvalues are $\lambda_k(\eta) = \eta^2 + \omega_k \eta^{4/3}$.

Remark 3.1. Let $\delta_{x=a}$ denote the Dirac distribution on \mathbb{R}_+ , $a > 0$, then it reads as follows:

$$(3.3) \quad \delta_{x=a} = \sum_{k \geq 1} e_k(x, \eta) e_k(a, \eta).$$

We define the gallery modes as follows:

Definition 3.2. For $x > 0$ let $E_k(\Omega)$ be the closure in $L^2(\Omega)$ of

$$(3.4) \quad \left\{ \frac{1}{2\pi} \int e^{iy\eta} e_k(x, \eta) \psi_k(\eta) d\eta, \hat{\psi}_k \in \mathcal{S}(\mathbb{R}) \right\},$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions,

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}), \|z^\alpha D^\beta f\|_{L^\infty(\mathbb{R})} < \infty \quad \forall \alpha, \beta \in \mathbb{N} \right\}.$$

For k fixed, a function in $E_k(\Omega)$ is called whispering gallery mode.

We have the following result (see [6]):

Theorem 3.3. *We have the orthogonal decomposition $(L^2(\Omega), \Delta_D) = \bigoplus_{\perp} E_k(\Omega)$, where $E_k(\Omega)$ denotes the space of gallery modes associated to the k th zero of the Airy function Ai and where $\Delta_D = \partial_x^2 + (1+x)\partial_y^2$ with Dirichlet boundary condition on $\partial\Omega$.*

Proof. Indeed, from [6][Section 2.2] one can easily see that $(E_k(\Omega))_k$ are closed, orthogonal and that $\cup_k E_k(\Omega)$ is a total family (i.e. that the vector space spanned by $\cup_k E_k(\Omega)$ is dense in $L^2(\Omega)$). \square

Let $\psi_j \in C_0^\infty(]0, \infty[)$ as in remark 1.10. Using remark 3.1, for $h \in (0, 1]$, we write the initial data, localized at frequency $\frac{1}{h}$, as follows

$$(3.5) \quad \psi_2(h\sqrt{-\Delta_g})\psi_1(hD_y)\delta_{x=a, y=0} = \sum_{1 \leq k} \frac{1}{2\pi h} \int e^{\frac{i}{h}y\eta} \psi_2(\eta\sqrt{1+\omega_k(h/\eta)^{2/3}}) \psi_1(\eta) e_k(a, \eta/h) e_k(x, \eta/h) d\eta.$$

Observe that in the sum over k , by remark 1.10 we may assume $k \leq \varepsilon h^{-1}$ with ε small. From (3.5) we get

$$(3.6) \quad u_{a,h}(t, x, y) = e^{-it\sqrt{-\Delta_g}} \psi_2(h\sqrt{-\Delta_g}) \psi_1(hD_y) \delta_{x=a, y=0} \\ = \sum_{1 \leq k} \frac{1}{2\pi h} \int e^{\frac{i}{h}(y\eta - t\eta\sqrt{1+\omega_k(h/\eta)^{2/3}})} \psi_2(\eta\sqrt{1+\omega_k(h/\eta)^{2/3}}) \psi_1(\eta) e_k(a, \eta/h) e_k(x, \eta/h) d\eta.$$

Our goal is to prove the following proposition.

Proposition 3.4. *There exists C such that for every $h \in]0, 1]$, every $0 < a \leq h^{1/2}$ and every $t \in [-1, 1]$, the following holds true*

$$(3.7) \quad \|\mathbb{1}_{x \leq a} u_{a,h}(t, x, y)\|_{L^\infty} \leq Ch^{-2} \min(1, h^{1/4} + (\frac{h}{|t|})^{1/3}).$$

This proposition will be proved in the next two sections. Proposition 3.4 clearly implies theorem 1.3 for $a \leq h^{1/2}$. By time symmetry, we may restrict ourselves to positive times $t \in [0, 1]$. Notice that the proof for the wave propagator $\exp(+it\sqrt{-\Delta_g})$ is exactly the same as the sign plays no role whatsoever.

3.2. Tangential initial directions. In this section, we make use of the Sobolev embedding properties related to the orthogonal basis (e_k) .

Lemma 3.5. *There exists C_0 such that for $L \geq 1$ the following holds true*

$$(3.8) \quad \sup_{b \in \mathbb{R}} \left(\sum_{1 \leq k \leq L} k^{-1/3} Ai^2(b - \omega_k) \right) \leq C_0 L^{1/3}.$$

Proof. From $|Ai(x)| \leq C(1 + |x|)^{-1/4}$, we get

$$J(b) = \sum_{1 \leq k \leq L} k^{-1/3} Ai^2(b - \omega_k) \lesssim \sum_{1 \leq k \leq L} k^{-1/3} \frac{1}{1 + |b - \omega_k|^{1/2}}.$$

From $\omega_k \simeq k^{2/3}$, we get easily with C independent of L and D large enough

$$\sup_{b \leq 0} J(b) \leq CL^{1/3}, \quad \sup_{b \geq DL^{2/3}} J(b) \leq CL^{1/3}.$$

Thus we may assume $b = L^{2/3}b'$ with $b' \in [0, D]$. Since $\omega_k = k^{2/3}g(k)$ with g being an elliptic symbol of degree 0, we are left to prove that

$$I(x) = L^{-1/3} \sum_{1 \leq k \leq L} (k/L)^{-1/3} \frac{1}{1 + L^{1/3}|x - (k/L)^{2/3}|^{1/2}}$$

satisfies $\sup_{x \in \mathbb{R}} I(x) \leq C_0 L^{1/3}$. Since we can split $[0, 1]$ into a finite union of intervals on which the function $\frac{t^{-1/3}}{1 + L^{1/3}|x - t^{2/3}|^{1/2}}$ is monotone, and since each term in the sum is bounded by 1, we get

$$I(x) \lesssim Cte + L^{2/3} \int_0^1 \frac{t^{-1/3}}{1 + L^{1/3}|x - t^{2/3}|^{1/2}} dt \leq Cte + L^{1/3} \int_0^1 \frac{3}{2|x - s|^{1/2}} ds,$$

and the proof of Lemma 3.5 is complete. \square

Let $u_{a,h,<L}$ be the function defined by (3.6) with the sum restricted to $k \leq L$. From (3.2), lemma 3.5, and Cauchy-Schwarz inequality, one gets

$$(3.9) \quad \|u_{a,h,<L}(t, x, y)\|_{L^\infty} \leq C_1 h^{-2} h^{1/3} L^{1/3}.$$

Taking $L = C/h$, we get that proposition 3.4 holds true for $|t| \leq h$. With $L = h^{-1/4}$ or $L = 1/|t|$, one sees also that (3.7) holds true for $u_{a,h,<L}$. Thus we are reduce to prove that (3.7) holds true for $u_{a,h,>L}$ which is defined by the sum over $k \geq L$ with $L \geq D \max(h^{-1/4}, 1/|t|)$ with a large constant D .

3.3. Non-tangential initial directions. In this section, we denote by $v_{a,h}(t, x, y)$ the function defined for $h \leq t \leq 1$ by (3.6) with the sum restricted to $L \leq k \leq \varepsilon/h$, $L \geq D \max(h^{-1/4}, 1/t)$, $D > 0$ large and $\varepsilon > 0$ small. For each value of k , we set

$$(3.10) \quad \lambda = t\omega_k h^{-1/3}, \quad \mu = \frac{ah^{-1/3}}{t\omega_k^{1/2}}.$$

From $\omega_k \simeq k^{2/3}$, $k \geq 1/t$, and $t \geq h$, one has for some $c > 0$ $\lambda \geq c$; thus we will take λ as our large parameter. However, the parameter μ just satisfies

$$(3.11) \quad 0 \leq \mu \lesssim \frac{h^{1/6}}{t} \min(t^{1/3}, h^{1/12})$$

and thus may be small or arbitrary large. Observe that for D large enough, for $k \geq Dh^{-1/4}$ and $0 \leq x \leq a \leq h^{1/2}$ one has

$$\omega_k - xh^{-2/3}\eta^{2/3} \geq \omega_k/2$$

for all η in the support of ψ_1 . Therefore we can use the asymptotic expansion of the Airy function from section 2.2, with $\omega = e^{i\pi/4}$,

$$Ai(\zeta) = \sum_{\pm} \omega^{\pm 1} e^{\mp \frac{2}{3}i(-\zeta)^{3/2}} (-\zeta)^{-1/4} \Psi_{\pm}(-\zeta)$$

which is valid for $-\zeta > 1$, with $-\zeta = \omega_k - \eta^{2/3}h^{-2/3}x \geq \omega_k/2 \geq \frac{\omega_1}{2} > 1$ since $\omega_1 \simeq 2, 33$. We thus get from (3.6)

$$(3.12) \quad v_{a,h} = \sum_{L \leq k \leq \varepsilon/h} w_k, \quad w_k = \frac{1}{2\pi h} \sum_{\pm, \pm} \int e^{i\lambda \Phi_k^{\pm, \pm}} \sigma_k^{\pm, \pm} d\eta$$

The phases $\Phi_k^{\pm, \pm}$ and the symbols $\sigma_k^{\pm, \pm}$ of w_k read as follows, with the notation $z = h^{2/3}\omega_k\eta^{-2/3} \geq 2a$:

$$(3.13) \quad h\lambda \Phi_k^{\pm, \pm}(t, x, y, \eta, a) = \eta \left(y - t\sqrt{1+z} \mp \frac{2}{3}(z-x)^{3/2} \mp \frac{2}{3}(z-a)^{3/2} \right),$$

$$(3.14) \quad \sigma_{k,\pm}^{\pm}(x, \eta, a, h) = h^{-1/3} \eta \psi_1(\eta) \psi_2(\eta \sqrt{1+z}) \frac{f_k^2}{k^{1/3}} \omega^{\pm} \omega^{\pm} \\ \times (z-x)^{-1/4} (z-a)^{-1/4} \Psi_{\pm}(\eta^{2/3}h^{-2/3}(z-x)) \Psi_{\pm}(\eta^{2/3}h^{-2/3}(z-a)).$$

One has $3\eta\partial_{\eta} = -2z\partial_z$ and for $0 \leq x \leq a \leq 2z$,

$$|(z\partial_z)^j((z-x)^{-1/4})| \leq C_j z^{-1/4} \leq C'_j(hk)^{-1/6};$$

moreover, Ψ_{\pm} are classical symbols of degree 0 at infinity and

$$|\eta^{2/3}h^{-2/3}(z-x)| \geq \omega_k/2 \geq Ch^{-1/6},$$

since $k \geq L \geq Dh^{-1/4}$. Therefore we get from (3.14) that for all j , there exists C_j independent of h, k, a, x, η such that

$$(3.15) \quad |\partial_{\eta}^j \sigma_k^{\pm, \pm}(x, \eta, a, h)| \leq C_j(hk)^{-2/3}.$$

Proposition 3.6. *For ε small, there exists C independent of $a \in (0, h^{1/2}]$, $t \in [h, 1]$, $x \in [0, a]$, $y \in \mathbb{R}$ and $k \in [L, \varepsilon/h]$ such that the following holds true*

$$(3.16) \quad \left| \int e^{i\lambda \Phi_k^{\pm, \pm}} \sigma_k^{\pm, \pm} d\eta \right| \leq C(hk)^{-2/3} \lambda^{-1/3}.$$

Observe that from (3.12) and the definition (3.10) of λ , (3.16) implies

$$(3.17) \quad \|\mathbb{1}_{x \leq a} v_{a,h}(t, x, y)\|_{L^{\infty}} \leq Ch^{-1} \sum_{k \leq \varepsilon/h} (hk)^{-2/3} t^{-1/3} h^{1/9} k^{-2/9} \\ = Ch^{-2} \left(\frac{h}{t}\right)^{1/3} h^{1/9} \left(\sum_{k \leq \varepsilon/h} k^{-8/9} \right) \leq C' h^{-2} \left(\frac{h}{t}\right)^{1/3}$$

and therefore proposition 3.4 holds true for $v_{a,h}$.

Proof. Since from (3.15) the $(hk)^{2/3}\sigma_k^{\pm,\pm}$ are classical symbols of degree 0 compactly supported in η , we intend to apply the stationary phase to an integral of the form

$$(3.18) \quad J = \int e^{i\lambda\Phi_k^{\pm,\pm}} g d\eta$$

with g a classical symbol of degree 0 compactly supported in η . We have to prove uniformly with respect to the parameters the inequality

$$(3.19) \quad |J| \leq C\lambda^{-1/3}.$$

Derivative of the phase with respect to η yields

$$h\lambda\partial_\eta\phi_k^{\pm,\pm} = y - t\frac{1+\frac{2}{3}z}{\sqrt{1+z}} \pm \frac{2}{3}x(z-x)^{1/2} \pm \frac{2}{3}a(z-a)^{1/2},$$

where the two \pm signs are independent from each other (thus, we have 4 cases to consider). Let $\delta = \frac{x}{a} \in [0, 1]$, $\alpha = \frac{a}{h^{2/3}\omega_k}$ and $s = \eta^{-2/3} \in [s_0, s_1]$. Since D is large, one has $\alpha \in [0, c_0]$ with c_0 such that $\eta^{-2/3} = s \geq s_0 \geq 2c_0$ on the support of $\psi_1(\eta)$. Let $X = \frac{y-t}{t\omega_k h^{2/3}}$ and define the function $g(z)$ by

$$\frac{1+\frac{2}{3}z}{\sqrt{1+z}} = 1 + zg(z), \quad g(z) = \frac{1}{6} + \frac{z}{24} + O(z^2).$$

Then the derivative of the phase is equal to

$$(3.20) \quad \partial_\eta\phi_k^{\pm,\pm} = X - sg(h^{2/3}\omega_k s) + \frac{2}{3}\mu\theta^{\pm,\pm}, \quad \theta^{\pm,\pm} = \pm\delta(s-\delta\alpha)^{1/2} \pm (s-\alpha)^{1/2}.$$

We now study critical points. We take $s = \eta^{-2/3}$ as variable and we get

$$(3.21) \quad \begin{aligned} \partial_s\partial_\eta\phi_k^{\pm,\pm} &= -(g(z) + zg'(z)) + \frac{\mu}{3}\left(\pm\delta(s-\delta\alpha)^{-1/2} \pm (s-\alpha)^{-1/2}\right) \\ \partial_s^2\partial_\eta\phi_k^{\pm,\pm} &= -h^{2/3}\omega_k(2g'(z) + zg''(z)) - \frac{\mu}{6}\left(\pm\delta(s-\delta\alpha)^{-3/2} \pm (s-\alpha)^{-3/2}\right). \end{aligned}$$

Lemma 3.7. *For ε small enough, there exists $c > 0$ independent of $k \leq \varepsilon/h$ such that*

$$|\partial_s\partial_\eta\phi_k^{\pm,\pm}| + |\partial_s^2\partial_\eta\phi_k^{\pm,\pm}| \geq c.$$

Proof. One has $(s-\alpha)^{-1/2} \geq \delta(s-\delta\alpha)^{-1/2}$; for ε small, $z = h^{2/3}\omega_k s$ is small and thus $g(z) + zg'(z)$ is close to $\frac{1}{6}$. Thus we get

$$|\partial_s\partial_\eta\phi_k^{\pm,-}| \geq 1/10.$$

The derivative $\partial_s\partial_\eta\phi_k^{\pm,+}$ may vanish but in case $|\partial_s\partial_\eta\phi_k^{\pm,+}| \leq 1/100$, the first line of (3.21) implies

$$\frac{\mu}{3}(s-\alpha)^{-1/2} \geq 0.05.$$

The second line of (3.21) then gives a positive lower bound on $|\partial_s^2 \partial_\eta \phi_k^{+,+}|$. It remains to study $\phi_k^{-,+}$. For any function f , one has

$$(3.22) \quad f(s - \alpha) - \delta f(s - \delta\alpha) = (1 - \delta)f(s - \delta\alpha) - \int_0^{\alpha(1-\delta)} f'(s - \delta\alpha - t) dt.$$

Taking $f(t) = t^{-1/2}$, we thus find that

$$|\partial_s \partial_\eta \phi_k^{-,+}| \leq 1/100 \implies \mu(1 - \delta) \geq c > 0.$$

If one applies (3.22) with $f(t) = t^{-3/2}$, we then find that for ε small, the second line of 3.21 implies $|\partial_s^2 \partial_\eta \phi_k^{-,+}| \geq c/2$. The proof of Lemma 3.7 is complete. \square

From lemma 3.7 and 2.19, we get that proposition 3.6 holds true in the case where the parameter μ is bounded, since in that case all the derivatives of order ≥ 2 of the phase $\phi_k^{\pm,\pm}$ are bounded. It remains to study the case where μ is large.

In cases $(+, +)$ or $(-, -)$, and μ large, we can take as large parameter $\Lambda = \lambda\mu$. Since $(s - \alpha)^{-1/2} + \delta(s - \delta\alpha)^{-1/2} \geq c > 0$, we get in that case that (3.16) holds true with a better factor $(hk)^{-2/3} \Lambda^{-1/2}$ on the right hand side.

It remains to study the cases $(+, -)$ and $(-, +)$ for μ large. But in that cases, we can use (3.22): therefore, if $\mu(1 - \delta)$ is bounded, all the derivatives of order ≥ 2 of the phase $\phi_k^{\pm,\mp}$ are bounded, and therefore from Lemmata 3.7 and 2.19, we get that proposition 3.6 holds true.

Finally, in the cases $(+, -)$ and $(-, +)$ and $\mu(1 - \delta)$ large, we can take as large parameter $\Lambda' = \lambda\mu(1 - \delta)$, and since by (3.22) one has

$$|(s - \alpha)^{-1/2} - \delta(s - \delta\alpha)^{-1/2}| \geq c(1 - \delta),$$

with $c > 0$, we get in that case that (3.16) holds true with a better factor $(hk)^{-2/3} \Lambda'^{-1/2}$ on the right hand side.

The proof of Proposition 3.6 is now complete. \square

This concludes the proof of Proposition 3.4.

4. DIMENSION $d \geq 3$

Let $d \geq 3$ and $\Omega_d = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}\}$ with Laplace operator $\Delta_d = \partial_x^2 + (1 + x)\Delta_y$. The normal variable is still denoted $x > 0$, and the boundary is still defined by the condition $x = 0$. The proof of Theorems 1.3 and 1.4 follows exactly the same lines as in the $2d$ case, both in the case $a \lesssim h^{1/2}$ and $a \gg h^{4/7}$.

4.1. Parametrix for $a \gg h^{4/7}$. In higher dimensions the parametrix construction is identical to the one in the two dimensional case. We set $\hbar = h/|\eta|$, and we define $v(t, x, y, h)$

with $y \in \mathbb{R}^{d-1}$ by

$$(4.1) \quad \begin{aligned} v(t, x, y, h) &= \sum_{0 \leq N \leq C_0/\sqrt{a}} v_N(t, x, y, h) \\ v_N(t, x, y, h) &= \frac{1}{(2\pi h)^d} \int e^{i\frac{2}{h}y} u_N(t, x, h/|\eta|) |\eta| \chi_0(|\eta|) d\eta, \end{aligned}$$

with the same $u_N(t, x, h/|\eta|)$ as before. We take polar coordinates in $\eta \in \mathbb{R}^{d-1}$, $\eta = |\eta|\omega$. We thus get

$$(4.2) \quad v_N(t, x, y, h) = \frac{1}{(2\pi h)^d} \int \left(\int e^{i\frac{|\eta|}{h}(y \cdot \omega)} d\omega \right) u_N(t, x, h/|\eta|) |\eta|^d \chi_0(|\eta|) d|\eta|,$$

In the above formula, apart from the harmless factor $|\eta|^d$ instead of $|\eta|$, we have a superposition with respect to $\omega \in S^{d-2}$ of functions of the same type as before, which are evaluated at $z = y \cdot \omega$. We shall use the following lemma.

Lemma 4.1. *Let $\psi_j \in C_0^\infty(]0, \infty[)$. There exists $c_0 > 0$ such that for every $a \in]0, 1]$ and every $t \in [h, 1]$*

$$(4.3) \quad h^d |\psi_1(h\sqrt{-\Delta_d}) \psi_2(h|D_y|) e^{\pm it\sqrt{-\Delta_d}} \delta_{x=a, y=0}|_{L^\infty(x \leq a, |y| \leq c_0 t)} \in O\left(\left(\frac{h}{t}\right)^\infty\right)$$

Proof. We may and will assume $a \leq 2t$; In fact, for $t \leq a/2$, by finite speed of propagation, the singular support of $e^{\pm it\sqrt{-\Delta_d}} \delta_{x=a, y=0}$ has not reached the boundary $x = 0$, and then (4.3) is a simple consequence of propagation of singularities in the interior (see the argument below). Let $T \in [h, 1]$ be given; perform the change of variable $t = Ts$, $x = TX$, $y = TY$, and set $f_T(s, X, Y) = f(Ts, TX, TY)$. Then one has

$$(\Delta_d f)_T = T^{-2} P_T f_T, \quad P_T = \partial_X^2 + (1 + TX) \Delta_Y$$

Set $\hbar = h/T \leq 1$. One has for any ψ the identity $(\psi(\hbar D_{t,x,y}) f)_T = \psi(\hbar D_{s,X,Y}) f_T$, and therefore (4.3) is equivalent to the estimate at time $s = 1$

$$(4.4) \quad |\psi_1(\hbar\sqrt{P_T}) \psi_2(\hbar|D_Y|) e^{\pm i\sqrt{-P_T}} \delta_{X=a/T, Y=0}|_{L^\infty(X \leq a/T, |Y| \leq c_0)} \in O(\hbar^\infty).$$

Observe that $b = a/T \leq 2$ is bounded. Since $\psi_2(\hbar|D_Y|)$ commutes with the flow $e^{\pm i\sqrt{-P_T}}$, using the Melrose-Sjöstrand theorem on propagation of singularities at the boundary [10], we just need to verify the following : There exists $c_0 > 0$ such that for any $T \in [0, 1]$ and any optical ray $s \rightarrow \rho(s)$ associated to the symbol $\xi^2 + (1 + TX)\eta^2$ starting at $t = 0$ from $\rho(0) = (X = b, Y = 0; \xi_0, \eta_0)$ with $\xi_0^2 + (1 + Tb)\eta_0^2 = 1$ and $|\eta_0| \geq c_1 > 0$, one has $|Y(\rho(1))| \geq 4c_0$. But on the generalized bicharacteristic flow, one has $\partial_s \eta = 0$ and $\partial_s Y = 2\eta(1 + TX(s))$ and therefore $Y(s) = \eta_0(g(s))$ with $g(s) \geq 2s$ and the result is obvious. Observe that the cutoff by $\psi_2(\hbar|D_Y|)$ is essential to get the lower bound on $|\eta_0|$. The proof of Lemma 4.1 is complete. \square

In order to prove our dispersive estimates, we may assume $h \leq t \leq 1$, and therefore by lemma 4.1, we may also assume $|y| \geq c_0 t \geq c_0 h$. Classical stationary phase in $\omega \in S^{d-2}$ gives

$$(4.5) \quad \int e^{i\frac{|\eta|}{h}(y \cdot \omega)} d\omega = \left(\frac{h}{|\eta||y|}\right)^{\frac{d-2}{2}} \left(e^{i\frac{|\eta||y|}{h}} \sigma_+ \left(\frac{h}{|\eta||y|}\right) + e^{-i\frac{|\eta||y|}{h}} \sigma_- \left(\frac{h}{|\eta||y|}\right) \right)$$

where σ_{\pm} are classical symbols of degree 0 in the small parameter $\frac{h}{|\eta||y|}$. Inserting (4.5) in (4.2), and since for $|y| \geq c_0 t$ and $|\eta| \in [\frac{1}{8}, 8]$ one has $(\frac{h}{|\eta||y|})^{\frac{d-2}{2}} \leq C(h/t)^{\frac{d-2}{2}}$, we easily see that the proof of theorem 1.3 and 1.4 follows exactly like in the $2d$ case.

4.2. Case $a \lesssim h^{1/2}$. Indeed, the dispersive estimates follow once we notice that definition 3.2 and theorem 3.3 extend to the d dimensional domain Ω_d . It is enough to define for $x > 0$, $E_k(\Omega_d)$ to be the closure in $L^2(\Omega_d)$ of

$$(4.6) \quad \left\{ \frac{1}{(2\pi)^{d-1}} \int e^{i\langle y, \eta \rangle} Ai(|\eta|^{\frac{2}{3}} x - \omega_k) \hat{\varphi}(\eta) d\eta, \varphi \in \mathcal{S}(\mathbb{R}^{d-1}) \right\},$$

where $\mathcal{S}(\mathbb{R}^{d-1})$ is the Schwartz space of rapidly decreasing functions,

$$\mathcal{S}(\mathbb{R}^{d-1}) = \left\{ f \in C^\infty(\mathbb{R}^{d-1}) \mid \|z^\alpha D^\beta f\|_{L^\infty(\mathbb{R}^{d-1})} < \infty \quad \forall \alpha, \beta \in \mathbb{N}^{d-1} \right\}.$$

Theorem 4.2. *We have the orthogonal decomposition $(L^2(\Omega_d), \Delta_d) = \bigoplus_{\perp} E_k(\Omega_d)$, where $E_k(\Omega_d)$ denotes the space of gallery modes associated to the k th zero of the Airy function Ai and where $\Delta_d = \partial_x^2 + (1+x)\Delta_y$ with Dirichlet boundary condition on $\partial\Omega_d$.*

Therefore, by lemma 4.1 and (4.5), the proof of our main theorems follows exactly like in the $2d$ case.

APPENDIX A. THE ENERGY CRITICAL NONLINEAR WAVE EQUATION

We consider the equation

$$(A.1) \quad \square_g u + |u|^{\frac{4}{d-2}} u = 0$$

with data $(u_0, u_1) \in H_0^1(\Omega_d) \times L^2(\Omega_d)$, with $3 \leq d \leq 6$. When the domain is \mathbb{R}^d , there is a long line of seminal works regarding this model, which may be one of the simplest model of a critical wave equation. To our knowledge, the first work to address the energy setting (as opposed to C^∞) is [12], where low dimensions are dealt with, using only the oldest Strichartz estimates (time and space exponents are equal). Higher dimensions (e.g. $d \geq 7$) have their own set of difficulties, mostly related to the low power nonlinearity $(1 + 4/(d-2)) < 2$ and the subsequent failure of its derivative with respect to u to be Lipschitz. All these technical annoyances may be solved one way or another, but are out of the scope of the present paper.

Hence we contend ourselves with the low dimensions. There are essentially two things to be checked:

- we have a “good” local Cauchy theory, providing energy class solutions ;

- this local Cauchy theory may be tweaked as to insert (a small power of) the potential energy of the solution in the nonlinear estimates, so that we can then perform the non concentration argument from [3] and extend our solutions globally in time. Remark that the potential energy is $\|u\|_{L_x^{2d/(d-2)}}^{2d/(d-2)}$ which corresponds to the critical nature of the equation, as by Sobolev embedding, $H_0^1 \hookrightarrow L^{2d/(d-2)}$.

We refer to [7], [8] for details on how to deal with fractional derivatives, Besov spaces on domains and product-type estimates (alternatively, one may proceed with interpolation as in [3]).

Remark A.1. Note that in proving Theorem 1.6 from Theorem 1.3, one needs, for $p > 2$, an embedding $\dot{B}_p^{0,2} \subset L^p$ on domains, which may be proved directly or follows from a Mikhlin-Hörmander multiplier theorem from Alexopoulos (see [7] and references therein).

Having these tools at hand, we may proceed exactly as in \mathbb{R}^d , provided we have the right set of exponents.

- Case $d = 3$: Theorem 1.6 allows for the Strichartz triplet $(q = 4, r = 12, \beta = 1)$ and one may proceed like in the \mathbb{R}^3 case. This was already observed in [2] and allows for a streamlined argument when compared to [3].
- Case $d = 4$: Theorem 1.6 allows for the Strichartz triplet $(q = 11/5, r = 22/3, \beta = 1)$. As by Sobolev embedding we have $H_0^1 \hookrightarrow L_x^4$, we may write

$$|u|^2 u \leq |u|^{4/5} |u|^{11/5} \in L_t^\infty L_x^5 \times L_t^1 L_x^{5/3} \subset L_t^1 L_x^2,$$

and we may proceed as in \mathbb{R}^4 .

- Case $d = 5$: Theorem 1.6 allows for the Strichartz triplet $(q = 2, r = 5, \beta = 1)$. As by Sobolev embedding we have $H_0^1 \hookrightarrow L_x^{10/3}$, we may write

$$|u|^{4/3} u \leq |u|^2 |u|^{1/3} \in L_t^1 L_x^{5/2} \times L_t^\infty L_x^{10} \subset L_t^1 L_x^2,$$

and we may proceed as in \mathbb{R}^5 .

- Case $d = 6$: Theorem 1.6 allows for the Strichartz triplet $(q = 2, r = 18/5, \beta = 1)$. By Sobolev embedding $\dot{B}_{18/5}^{1/3,2} \hookrightarrow L_x^4$, we get $u \in L_t^2 L_x^4$ which provides a local Cauchy theory but without the potential energy factor. However we may estimate

$$|u|u \in L_t^2 L_x^4 \times L_t^\infty \dot{B}_{36/17}^{2/3,2} \subset L_t^2 \dot{B}_{18/13}^{2/3,2},$$

which is the dual endpoint Strichartz space. As we may estimate the $\dot{B}_{36/17}^{2/3,2}$ norm of u in term of H_0^1 and L_x^3 norms, we now have a good local Cauchy theory, suitable to globalization in time.

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